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CONTENTS

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ROY DUBISCH, *Editor*

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EDITORIAL

At the time of assuming the editorship of the MATHEMATICS MAGAZINE it seems appropriate for me to state once again the objectives of this publication.

The mathematical level of the MATHEMATICS MAGAZINE has been set by the Publications Committee of the Mathematical Association of America as lying somewhere between the *American Mathematical Monthly* and the *Mathematics Teacher*. We are trying to publish a wide variety of mathematical material that will be of interest to undergraduate students and to teachers at both the college and secondary levels.

We are interested in publishing articles giving new insights into well-known problems, in expository articles at various levels, in articles on the teaching of mathematics, the history and philosophy of mathematics, and in some of the incidentals of mathematics such as references to mathematics as used by non-mathematicians, humorous aspects of mathematics, etc. We are also interested in publishing research papers providing that the understanding of the paper does not demand a high level of mathematical sophistication and that the topic is not unduly specialized. Our Problems and Questions department attempts to propose problems at a wide variety of levels of difficulty.

In brief, the MATHEMATICS MAGAZINE is dedicated to the presentation of good collegiate level mathematics in a simple and understandable way. The editors welcome your manuscripts that will help us achieve this goal. We will also welcome criticism and suggestions for the improvement of the journal from our readers.

Readers will note that this issue of MATHEMATICS MAGAZINE does not have separate sections on the teaching of mathematics and comments on papers and books. This does *not* mean that articles on the teaching of mathematics and comments on papers are not welcome but simply that they will become a part of the entire journal. Frequently, too, it is difficult to decide whether an article is primarily of pedagogic interest. In later issues book reviews may be reinstated as a separate section. The principal problem has been lack of space and duplication of effort with other journals in covering a wide range of books. It is quite possible that arrangements will be made later to distinguish the kinds of books to be reviewed in the MATHEMATICS MAGAZINE, the *American Mathematical Monthly*, and the *Mathematics Teacher*.

ROY DUBISCH

AN ELEMENTARY DERIVATION OF THE CAUCHY, HÖLDER, AND MINKOWSKI INEQUALITIES FROM YOUNG'S INEQUALITY

ELMER TOLSTED, Pomona College

1. Introduction. Three of the most famous "classical inequalities" are those of Cauchy, Hölder, and Minkowski. These inequalities are "pulled out of the hat" so frequently in mathematical proofs that an early acquaintance with them would be useful for most students.

We shall deduce these three inequalities from an inequality involving integrals due to W. H. Young. In section 2 we present an elementary geometric proof of Young's inequality. In section 3 we obtain an important special case of Young's inequality from which we deduce the Hölder and Minkowski inequalities for finite sequences in sections 4 and 5. Finally in section 6 we use the inequality in section 3 to prove Hölder's and Minkowski's inequalities for Riemann integrals of continuous functions.

In our presentation Cauchy's inequality appears simply as a special case of Hölder's inequality. Historically, Cauchy's inequality was published in 1821, whereas Hölder's generalization did not appear until 1889. Minkowski's inequality appeared in 1896, while Young's inequality, which we use as a point of departure, was not published until 1912. See the References for details.

This approach seems desirable for an elementary discussion for several reasons:

First: All three of these famous inequalities are achieved easily and quickly. Moreover, the geometrically obvious necessary and sufficient condition for equality to hold in the case of Young's inequality leads directly to necessary and sufficient conditions for equality in the other cases.

Second: The Hölder and Minkowski inequalities are demonstrated for real exponents. The usual extra limiting process needed to proceed from rational to real exponents is avoided because of the elementary calculus assumed in applying Young's inequality.

Third: There is an appealing novelty in deducing inequalities for sequences from one containing integrals. This is the reverse of the usual procedure in intermediate courses in analysis, where sequences are studied first and properties of integrals deduced later.

Note 1. In his two-volume work, *Trigonometric Series*, Zygmund begins the section on inequalities by assuming Young's inequality as geometrically obvious. He then deduces from it, in a most elegant and condensed fashion, the above inequalities together with many others for finite and infinite sequences of complex numbers and for Lebesgue integrals.

Note 2. In the book entitled *Inequalities*, by Hardy, Littlewood, and Polya, there are 404 theorems each containing one or more inequalities which are frequently used by mathematicians. The inequalities of Young, Hölder, Cauchy, and Minkowski appear as Theorems 156, 11, 7, and 25 respectively. Obviously this approach is considerably different from that of Zygmund. We are indebted to this comprehensive work for much of our bibliography.

Note 3. *Analytic Inequalities*, by Kazarinoff gives a most readable introduction to inequalities for the undergraduate student. We recommend this 90-page book most highly. The Hölder and Minkowski inequalities are deduced on pages 67–74.

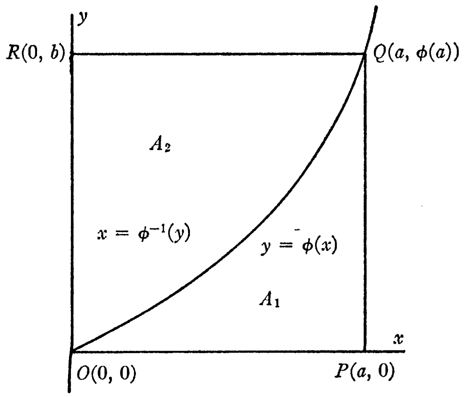

FIG. 1. $b=\phi(a)$, $ab=A_1+A_2$.

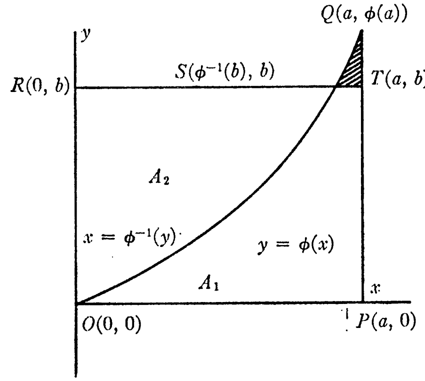

FIG. 2. $b<\phi(a)$, $ab<A_1+A_2$.

2. **Young's inequality.** Let $y=\phi(x)$ be a continuous, strictly increasing function for $x \geqq 0$, and let $\phi(0)=0$ and $\phi(a)=b$, where a and b are any positive real numbers. (See Figure 1.) Solving this equation for x in terms of y we obtain $x=\phi^{-1}(y)$. We call ϕ^{-1} the function inverse to ϕ, and it too is a continuous, strictly increasing function. Note that $\phi^{-1}(0)=0$ and $\phi^{-1}(b)=a$, and that the equations $y=\phi(x)$ and $x=\phi^{-1}(y)$ have the same graph.

From elementary calculus we know that the areas A_1 and A_2 in Figure 1 are given by

$$A_1 = \int_0^a y dx = \int_0^a \phi(x) dx \tag{2.1}$$

and

$$A_2 = \int_0^b x dy = \int_0^b \phi^{-1}(y) dy = \int_0^b \phi^{-1}(x) dx. \tag{2.2}$$

Since ab, the area of the rectangle $OPQR$ in Figure 1, is equal to the sum of the areas A_1 and A_2, it follows from (2.1) and (2.2) that

$$ab = \int_0^a \phi(x) dx + \int_0^b \phi^{-1}(x) dx. \tag{2.3}$$

Suppose now that b is not equal to $\phi(a)$. (See Figure 2 for the case $\phi(a)>b$ and Figure 3 for the case $\phi(a)<b$.) In each figure, ab is the area of rectangle $OPTR$. But this area is smaller than the sum of areas A_1 and A_2 by the amount STQ which is shaded in the figures.

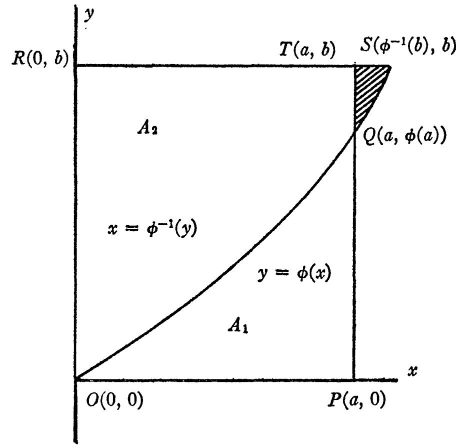

FIG. 3. $b>\phi(a)$, $ab<A_1+A_2$.

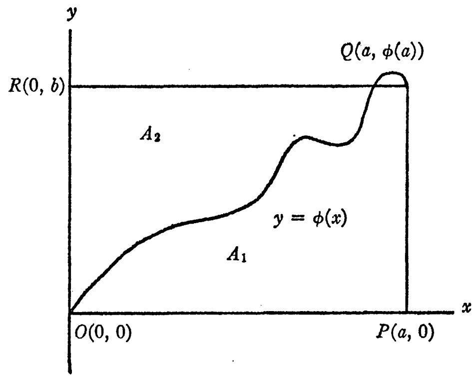

FIG. 4. $b=\phi(a)$, $ab\leqq A_1+A_2$.

Combining these three cases we obtain *Young's inequality* which states:

Let $\phi(x)$ and $\phi^{-1}(x)$ be two functions, continuous, vanishing at the origin, strictly increasing, and inverse to each other. Then for a, $b\geqq 0$ we have

$$ab \leqq \int_0^a \phi(x)dx + \int_0^b \phi^{-1}(x)dx. \tag{2.4}$$

From our previous discussion it is clear that equality holds if and only if

$$b = \phi(a). \tag{2.5}$$

Note 4. Had $\phi(x)$ not been assumed to be strictly increasing, but merely continuous, as suggested in Figure 4, we could not use the simple formula (2.2) for area A_2. Fortunately, in our applications of Young's inequality, the functions $\phi(x)$ will vanish at the origin and be continuous and strictly increasing for $x\geqq 0$.

3. Applications of Young's inequality. In this section we shall obtain inequalities from Young's inequality by choosing particular functions $\phi(x)$.

Example 1. Let $\phi(x)=x$. Then $\phi^{-1}(x)=x$ and (2.4) yields

$$ab \leqq \int_0^a xdx + \int_0^b xdx = \frac{a^2}{2} + \frac{b^2}{2}\cdot$$

This is the well-known inequality

$$2ab \leqq a^2 + b^2. \tag{3.1}$$

The condition $b=\phi(a)$ which is necessary and sufficient for equality in (3.1) is in this case

$$b = a. \tag{3.2}$$

Note 5. The usual way of proving (3.1) and (3.2) is simply to use the fact that $(a-b)^2\geqq 0$.

We now get the inequality essential to our purposes by choosing

$$\phi(x) = x^{\alpha} \qquad (\alpha > 0).$$

In this case the inverse function is $\phi^{-1}(x) = x^{1/\alpha}$, and (2.4) yields

$$ab \leqq \int_0^a x^{\alpha}dx + \int_0^b x^{1/\alpha}dx = \frac{a^{\alpha+1}}{\alpha+1} + \frac{b^{(1/\alpha)+1}}{(1/\alpha)+1}\cdot$$

If in this last inequality we let $r = \alpha+1$ and $r' = (1/\alpha)+1$, then we may write

$$ab \leqq \frac{1}{r}a^r + \frac{1}{r'}b^{r'}. \tag{3.3}$$

Since $[1/\{\alpha+1\}]+[1/\{(1/\alpha)+1\}] = 1$ we see that r and r' are related by $(1/r)+(1/r') = 1$ or equivalently

$$r' = \frac{r}{r-1}\cdot \tag{3.4}$$

Thus the inequality (3.3) holds under the conditions $a > 0$, $b > 0$, $r > 1$, and $(1/r)+(1/r') = 1$.

The condition for equality in (3.3) is derived again from (2.5). In this case the condition $b = \phi(a)$ becomes

$$b = a^{\alpha} = a^{r-1}. \tag{3.5}$$

Note 6. An even more roundabout way of obtaining (3.1) and (3.2) would be to let $r = r' = 2$ in (3.3) and (3.5).

Note 7. It is by allowing α to be any positive real exponent in the preceding integration that we are able to prove Hölder's inequality for real rather than rational exponents only in the next section. Note that r and r' may be any real numbers greater than unity and related by the formula in (3.4).

Note 8. Inequality (3.3) appears as Theorem 61 in Hardy, Littlewood, and Polya but is not there credited to any individual.

Note 9. If $a = 0$ and (or) $b = 0$ the reader can verify that (3.3) still holds. This fact will enable us to prove, in sections 4, 5, and 6, the Hölder and Minkowski inequalities in cases in which some terms of the sequences or some values of the functions involved are zero.

One could choose values for r in (3.3) such as 3, π, 5, sec 31°, etc., and obtain all kinds of nonobvious but also noninteresting inequalities. However, we shall use (3.3) in the next section to deduce Hölder's inequality which is also nonobvious but is not noninteresting.

4. Hölder's inequality. In order to keep printing and visual complexities at a minimum, we shall prove the three classical inequalities for finite sequences of positive real numbers only. The two sequences involved will be denoted by $\{a_i\} \equiv \{a_1, a_2, \cdots, a_n\}$, and $\{b_i\} \equiv \{b_1, b_2, \cdots, b_n\}$.

Note 10. We could prove our three inequalities for moduli of complex numbers by inserting appropriate absolute value signs throughout the following proofs. But this would clutter up the pages considerably. The following proofs are also valid for infinite sequences provided all sums involved are finite.

Let

$$S = (a_1^r + a_2^r + \cdots + a_n^r)^{1/r} \equiv (\sum a_i^r)^{1/r},$$
$$T = (b_1^{r'} + b_2^{r'} + \cdots + b_n^{r'})^{1/r'} \equiv (\sum b_i^{r'})^{1/r'}. \tag{4.1}$$

Replace a and b in (3.3) by a_i/S and b_i/T respectively. Then for $i=1, 2, \cdots, n$ we get the n inequalities

$$\frac{a_i}{S}\frac{b_i}{T} \leqq \frac{1}{r}\left(\frac{a_i}{S}\right)^r + \frac{1}{r'}\left(\frac{b_i}{T}\right)^{r'} \qquad (i = 1, 2, \cdots, n). \tag{4.2}$$

Adding up the right and left hand sides of the n inequalities (4.2) and, using (4.1) and (3.4), we get

$$\begin{aligned} \frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{ST} &\leqq \frac{1}{r}\left(\frac{a_1^r + a_2^r + \cdots + a_n^r}{S^r}\right) \\ &\quad + \frac{1}{r'}\left(\frac{b_1^{r'} + b_2^{r'} + \cdots + b_n^{r'}}{T^{r'}}\right) \\ &= \frac{1}{r}(1) + \frac{1}{r'}(1) = 1. \end{aligned} \tag{4.3}$$

Finally multiplying both extremes of (4.3) by ST, we have $a_1b_1+a_2b_2+\cdots+a_nb_n \leqq ST$ which is Hölder's inequality. Using the $\sum$ notation and (4.1), we write *Hölder's inequality* in the usual form,

$$\sum a_ib_i \leqq (\sum a_i^r)^{1/r}(\sum b_i^{r'})^{1/r'}, \tag{4.4}$$

where r and r' are real numbers greater than unity and $(1/r)+(1/r')=1$.

We shall have equality in Hölder's inequality if and only if we have equality in each of the n inequalities (4.2). This happens if and only if (3.5) is satisfied n times, that is if

$$\frac{b_i}{T} = \left(\frac{a_i}{S}\right)^{r-1} \qquad (i = 1, 2, \cdots, n). \tag{4.5}$$

Since S and T are independent of i, we see that (4.5) implies there is a constant k $(k=T/S^{r-1})$ such that

$$b_i = ka_i^{r-1} \qquad (i = 1, 2, \cdots, n). \tag{4.6}$$

In this case, we say that the sequences $\{a_i^{r-1}\}$ and $\{b_i\}$ are proportional. Conversely, if $\{a_i^{r-1}\}$ and $\{b_i\}$ are proportional, so that there is a constant k for which (4.6) holds, one can show that $k=T/S^{r-1}$ and thus that (4.6) implies

(4.5). This proves that equality holds in Hölder's inequality if and only if (4.6) is satisfied.

If we now raise both sides of (4.6) to the r' power, we have

$$b_i^{r'} = k^{r'} a_i^{(r-1)r'} = k^{r'} a_i^{r} \qquad (i = 1, 2, \cdots, n). \tag{4.7}$$

Since $k^{r'}$ is also a constant, we may conclude from (4.7) that Hölder's inequality becomes an equality if and only if the sequences $\{a_i^r\}$ and $\{b_i^{r'}\}$ are proportional.

If we now let $r=r'=2$ in (4.4) and (4.6), we have *Cauchy's inequality*,

$$\sum a_i b_i \leqq \left(\sum a_i^2\right)^{1/2} \left(\sum b_i^2\right)^{1/2} \tag{4.8}$$

with equality holding if and only if

$$b_i = k a_i. \tag{4.9}$$

Note 11. So far we have proved Cauchy's inequality for positive a_i and b_i. It is easy to see that the inequality will still hold if some or all of the a_i and b_i are allowed to be negative; for the right hand side will be unaffected by a change in sign of the a_i or b_i, and the left side will certainly not increase numerically. Thus Cauchy's inequality holds when a_i and b_i are any real numbers. See Note 9 for the case in which a_i or b_i assume the value zero.

5. Minkowski's inequality. For any real number $r>1$ we may write

$$\begin{aligned} \sum (a_i + b_i)^r &= \sum (a_i + b_i)(a_i + b_i)^{r-1} \\ &= \sum a_i(a_i + b_i)^{r-1} + \sum b_i(a_i + b_i)^{r-1}. \end{aligned} \tag{5.1}$$

Application of Hölder's inequality (4.4) to each of the two sums on the right hand side of (5.1) yields

$$\begin{aligned} \sum (a_i + b_i)^r \leqq \left[\sum a_i^r\right]^{1/r} \left[\sum (a_i + b_i)^{(r-1)r'}\right]^{1/r'} \\ + \left[\sum b_i^r\right]^{1/r} \left[\sum (a_i + b_i)^{(r-1)r'}\right]^{1/r'}. \end{aligned}$$

Factoring the right hand side and using (3.4) we have

$$\sum (a_i + b_i)^r \leqq \left[\sum (a_i + b_i)^r\right]^{1/r'} \left[\left(\sum a_i^r\right)^{1/r} + \left(\sum b_i^r\right)^{1/r}\right]. \tag{5.2}$$

Dividing both sides of (5.2) by the first factor on the right hand side gives us

$$\left[\sum (a_i + b_i)^r\right]^{1-(1/r')} \leqq \left(\sum a_i^r\right)^{1/r} + \left(\sum b_i^r\right)^{1/r}.$$

Noting that $1-(1/r')=1/r$, we write this as

$$\left[\sum (a_i + b_i)^r\right]^{1/r} \leqq \left(\sum a_i^r\right)^{1/r} + \left(\sum b_i^r\right)^{1/r} \qquad (r \geqq 1), \tag{5.3}$$

which is *Minkowski's inequality*.

Note 12. If $r=1$, (5.3) obviously becomes an equality. If $r<1$, the inequality is reversed. See Theorem 25 of Hardy, Littlewood, and Polya.

In order to find necessary and sufficient conditions for equality in Minkowski's inequality we note that Hölder's inequality was applied to each of the two sums on the right hand side of (5.1). To ensure equality in each of these applications of Hölder's inequality we must satisfy two sets of conditions of the type (4.6), namely:

$$(5.4)\qquad \begin{aligned} (a_i + b_i)^{r-1} &= k_1^{r-1} a_i^{r-1} \\ (a_i + b_i)^{r-1} &= k_2^{r-1} b_i^{r-1} \qquad (i = 1, 2, \cdots, n) \end{aligned}$$

where the role of k in (4.6) is played by k_1^{r-1} and k_2^{r-1} respectively. Extracting the $(r-1)$st roots in (5.4) gives us the equivalent conditions

$$(5.5)\qquad \begin{aligned} a_i + b_i &= k_1 a_i \qquad b_i = (k_1 - 1)a_i \\ &\text{or} \\ a_i + b_i &= k_2 b_i \qquad b_i = \{1/(k_2 - 1)\} a_i. \end{aligned}$$

We can satisfy the 2 sets of equations in (5.5) simultaneously if and only if $k_1 - 1 = \{1/(k_2 - 1)\}$. In this case we may write (5.5) as one set of n equations

$$(5.6)\qquad b_i = ka_i \qquad (i = 1, 2, \cdots, n),$$

where

$$(5.7)\qquad k = k_1 - 1 = \frac{1}{k_2 - 1}\cdot$$

Since (5.4) implies (5.6) we now know that (5.6) is necessary for equality in Minkowski's inequality. Question: Does (5.6) imply (5.5) and (5.4)? Answer: Yes, for if we are given two proportional sequences $\{a_i\}$ and $\{b_i\}$ and hence a value for k such that (5.6) is satisfied, we may use (5.7) to compute $k_1 = k + 1$ and $k_2 = (1/k) + 1$ and be sure that (5.5) and hence (5.4) are satisfied. Thus (5.6) is also sufficient to ensure equality in (5.3).

Note 13. For $r = 2$ we can give elementary geometric interpretations for the Hölder (Cauchy) and Minkowski inequalities. Let $n = 3$, and let $\boldsymbol{A}$ and $\boldsymbol{B}$ be vectors having the components $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ respectively. We know from elementary vector analysis that the lengths of vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $(\boldsymbol{A}+\boldsymbol{B})$ are given respectively by

$$|\boldsymbol{A}| = \sqrt{\left\{\sum_1^3 a_i^2\right\}}, \qquad |\boldsymbol{B}| = \sqrt{\left\{\sum_1^3 b_i^2\right\}}, \text{ and}$$

$$|\boldsymbol{A}+\boldsymbol{B}| = \sqrt{\left\{\sum_1^3 (a_i + b_i)^2\right\}}.$$

Thus the Minkowski inequality,

$$\sqrt{\left\{\sum_1^3 (a_i + b_i)^2\right\}} \leqq \sqrt{\left\{\sum_1^3 a_i^2\right\}} + \sqrt{\left\{\sum_1^3 b_i^2\right\}},$$

asserts the obvious geometric fact that

$$|A+B| \leqq |A| + |B|. \tag{5.8}$$

This is the so called *triangle inequality.*

We also know from elementary vector analysis that the scalar or "dot" product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, defined as $|A|\ |B|\ \cos\theta$, is equal to $\sum_1^3 a_i b_i$. (Here θ is the angle between the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.) Since $\cos\theta \leqq 1$ we know that

$$|A|\ |B|\ \cos\theta \leqq |A|\ |B|. \tag{5.9}$$

This is equivalent to Cauchy's inequality,

$$\sum_1^3 a_i b_i \leqq \sqrt{\left\{\sum_1^3 a_i^2\right\}} \sqrt{\left\{\sum_1^3 b_i^2\right\}}.$$

It is geometrically obvious that we shall have equality in (5.8) and (5.9) if and only if the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same direction. We know that two vectors have the same direction if and only if their components are proportional, which in this case means that

$$b_i = ka_i \qquad (i = 1, 2, 3).$$

We recognize this last equation as (4.9) and (5.6). This geometric argument has thus led us to our necessary and sufficient condition for equality to hold in both the Cauchy and the Minkowski inequalities for the case when $r=2$ and $n=3$.

6. Integral inequalities. We shall now state and prove the Cauchy, Hölder, and Minkowski inequalities for Riemann integrals of continuous functions.

If $f(x)$ and $g(x)$ are continuous, nonnegative functions on the closed interval $c \leqq x \leqq d$, then the following inequalities are true.

Cauchy-Schwarz:

$$\left[\int_c^d f(x)g(x)dx\right]^2 \leqq \left[\int_c^d f^2(x)dx\right]\left[\int_c^d g^2(x)dx\right]. \tag{6.1}$$

Equality holds if and only if

$$g(x) \equiv kf(x). \tag{6.2}$$

Hölder:

$$\int_c^d f(x)g(x)dx \leqq \left[\int_c^d f^r(x)dx\right]^{1/r}\left[\int_c^d g^{r'}(x)dx\right]^{1/r'}, \tag{6.3}$$

where $(1/r)+(1/r')=1$ and r and r' are real numbers greater than 1. Equality holds if and only if

$$g(x) \equiv k[f(x)]^{r-1}. \tag{6.4}$$

Minkowski:

$$\left[\int_c^d \{f(x)+g(x)\}^r dx\right]^{1/r} \leqq \left[\int_c^d f^r(x)dx\right]^{1/r} + \left[\int_c^d g^r(x)dx\right]^{1/r}, \tag{6.5}$$

where r is any real number greater than or equal to 1. The necessary and sufficient condition for equality is (6.2).

Note 14. Cauchy's inequality for integrals is called the Cauchy-Schwarz inequality and is again the special case of Hölder's inequality for which $r=r'=2$. It was first proved by Cauchy for finite sums, by Buniakowski for classical integrals, and by Schwarz for Lebesgue integrals.

Note 15. Since $f(x)$ and $g(x)$ are continuous nonnegative functions so are $f^r(x)$ and $g^{r'}(x)$; hence all the above Riemann integrals exist. The proofs that follow are valid for Lebesgue integrals provided all integrals exist. Equality will hold for Lebesgue integrals if the conditions for equality stated in (6.2) and (6.4) hold for almost all x (instead of for all x) on $c \leqq x \leqq d$.

Our proof of (6.3) is strictly analogous to the proof of Hölder's inequality in section 4. Assume that neither $f(x)$ nor $g(x)$ is identically zero on $c \leqq x \leqq d$, and let

$$S = \left(\int_c^d f^r(x)dx\right)^{1/r} \quad \text{and} \quad T = \left(\int_c^d g^{r'}(x)dx\right)^{1/r'}. \tag{6.6}$$

Since $S \neq 0$ and $T \neq 0$, we may choose $a=f(x)/S$ and $b=g(x)/T$ in (3.3) obtaining

$$\frac{f(x)}{S}\,\frac{g(x)}{T} \leqq \frac{1}{r}\,\frac{f^r(x)}{S^r} + \frac{1}{r'}\,\frac{g^{r'}(x)}{T^{r'}} \qquad (c \leqq x \leqq d). \tag{6.7}$$

Since S and T are definite integrals, they are constants; hence we may factor them out from under the integral signs when we integrate both sides of (6.7). This yields

$$\begin{aligned} \frac{1}{ST}\int_c^d f(x)g(x)dx &\leqq \frac{1}{r}\left[\frac{\int_c^d f^r(x)dx}{S^r}\right] + \frac{1}{r'}\left[\frac{\int_c^d g^{r'}(x)dx}{T^{r'}}\right] \\ &= \frac{1}{r}[1] + \frac{1}{r'}[1] = 1. \end{aligned} \tag{6.8}$$

Multiplying (6.8) by ST yields $\int_c^d f(x)g(x)dx \leqq ST$ which, in view of (6.6), is Hölder's inequality (6.3).

Question: Are we sure that integrating both sides of an inequality like (6.7) really preserves the inequality yielding the inequality (6.8)? *Answer*: Yes, for if we regard the left side of (6.7) as a continuous function $\phi(x)$ and the right hand side of (6.7) as a continuous function $\psi(x)$, we may call upon the following theorem from elementary calculus:

Let $\phi(x)$ and $\psi(x)$ be continuous functions satisfying the inequality $\phi(x) \leqq \psi(x)$ for all x on $c \leqq x \leqq d$. Then we have

$$\int_c^d \phi(x)dx \leqq \int_c^d \psi(x)dx. \tag{6.9}$$

Moreover, equality holds in (6.9) *if and only if*

$$(6.10) \qquad \phi(x) = \psi(x) \qquad \textit{for all } x \textit{ on } c \leqq x \leqq d.$$

This theorem is obvious if interpreted in terms of areas.

From (6.9) and (6.10) we now know that our integrals are equal when and only when their integrands are equal for all x on $c \leqq x \leqq d$. Thus we have equality in Hölder's inequality if and only if equality holds in (6.7) for all x on $c \leqq x \leqq d$. It then follows from (3.5) that

$$(6.11) \qquad \frac{g(x)}{T} = \left[\frac{f(x)}{S}\right]^{r-1} \qquad \text{for all } x \text{ on } c \leqq x \leqq d$$

is a necessary and sufficient condition for equality in (6.3). Since S and T are constants, (6.11) is equivalent to (6.4).

To prove Minkowski's inequality for integrals, the reader must go through the same procedures with integrals as we did with sums in section 5. Substituting $k\,f(x)$ for $g(x)$ in both sides of (6.5) and showing that each side reduces to $(1+k)\left[\int_c^d f^r(x)dx\right]^{1/r}$ is the easy way to show that $g(x) = k\,f(x)$ is sufficient to ensure equality. An argument analogous to that leading to (5.6) would show that $g(x) \equiv k\,f(x)$ is also necessary for equality in Minkowski's inequality.

Note 16. If we allow $f(x)$ and $g(x)$ to be discontinuous, $\phi(x)$ and $\psi(x)$ in (6.9) may also be discontinuous. In this case the integrals in (6.9) will remain equal even though $\phi(x) \neq \psi(x)$ at a denumerable set of values of x on $c \leqq x \leqq d$. Thus we are assured of equality in (6.1), (6.3), (6.5) if the condition stated for equality in each case holds for all but a denumerable set of values of x on $c \leqq x \leqq d$.

Note 17. We could have proved each of the three inequalities for integrals from the corresponding inequality for sequences, by approximating the integrals by step functions and using the corresponding inequality for sequences on the step functions. To write out such proofs in detail is much more lengthy and cumbersome than our method.

Note 18. A very condensed derivation of the Hölder, Cauchy-Schwarz, and Minkowski inequalities from Young's inequality is presented on pages 20–22 of Volume 1, *Metric and Normed Spaces*, of the two-volume work, *Elements of the Theory of Functions and Functional Analysis*, by Kolmogorov and Fomin. These inequalities are there crucial in proving that the distance, $\rho(x, y)$, between points x and y of the authors' numerous examples of metric spaces satisfies the *triangle inequality*:

$$\rho(x, y) + \rho(y, z) \geqq \rho(x, z).$$

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A REMARK CONCERNING THE DEFINITION OF A FIELD

A. H. LIGHTSTONE, Carleton University (Canada)

A convenient and well-known method of defining a field utilizes the notion of an abelian group. Thus, by a field we mean any algebraic system, say $(S, +, \cdot, 0, 1)$, where $+$ and $\cdot$ are binary operators on S, $0 \in S$ and $1 \in S$, which possesses the following properties.

(i) $(S, +, 0)$ is an abelian group.

(ii) $(S - \{0\}, \cdot', 1)$ is an abelian group, where $\cdot'$ is $\cdot$ restricted to $S - \{0\}$.

(iii) The following propositions are true about $(S, +, \cdot, 0, 1)$:

$$\text{(a) } 0 \neq 1$$

$$\text{(b) } \forall x \forall y \forall z [x \cdot (y + z) = x \cdot y + x \cdot z]$$

$$\text{(c) } \forall x \forall y \forall z [(y + z) \cdot x = y \cdot x + z \cdot x].$$

At first sight, it might appear that (c) is unnecessary. The purpose of this note is to demonstrate that (c) is *not* a consequence of the other postulates.

Consider the algebraic system $(\{0, 1\}, +, \cdot, 0, 1)$, where $+$ and $\cdot$ are defined by the following tables:

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	1
1	0	1

Clearly, $(\{0, 1\}, +, 0)$ is an abelian group, and $(\{1\}, 1)$, is an abelian group. Furthermore, the left-hand distributive law holds, since $\forall x \forall y [x \cdot y = y]$ holds in the given algebraic system. However, the right-hand distributive law fails, since $(1+1) \cdot 1 = 1$ while $1 \cdot 1 + 1 \cdot 1 = 0$. It follows that (c) is not a consequence of the remaining field postulates.

To clarify the intended meaning of postulates (i), (ii), and (iii), we spell out these statements in terms of propositions about the algebraic system $(S, +, \cdot, 0, 1)$.

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To clarify the intended meaning of postulates (i), (ii), and (iii), we spell out these statements in terms of propositions about the algebraic system $(S, +, \cdot, 0, 1)$.

(1) $\forall x \forall y \forall z[x+(y+z)=(x+y)+z]$

(2) $\forall x[x+0=x]$

(3) $\forall x \exists y[x+y=0]$

(4) $\forall x \forall y[x+y=y+x]$

(5) $\forall x \forall y \forall z[x \neq 0 \wedge y \neq 0 \wedge z \neq 0 \rightarrow x \cdot(y \cdot z)=(x \cdot y) \cdot z]$

(6) $\forall x[x \neq 0 \rightarrow x \cdot 1=x]$

(7) $\forall x \exists y[x \neq 0 \rightarrow x \cdot y=1 \wedge y \neq 0]$

(8) $\forall x \forall y[x \neq 0 \wedge y \neq 0 \rightarrow x \cdot y=y \cdot x]$

(9) $\forall x \forall y[x \neq 0 \wedge y \neq 0 \rightarrow x \cdot y \neq 0]$

(10) $0 \neq 1$

(11) $\forall x \forall y \forall z[x \cdot(y+z)=x \cdot y+x \cdot z]$

(12) $\forall x \forall y \forall z[(y+z) \cdot x=y \cdot x+z \cdot x]$.

Note that (1)–(4) state that $(S, +, 0)$ is an abelian group, and (5)–(9) state that $(S-\{0\}, \cdot', 1)$ is an abelian group. We are assuming, of course, that $+$ and $\cdot$ are binary operators on S; hence the corresponding postulate for a group is not required.

These twelve postulates are easily reduced to the usual ten, by first demonstrating that $\forall x[x \cdot 0=0]$ and $\forall x[0 \cdot x=0]$. These follow from the two distributive laws. It is easily shown that (9) is a consequence of the other postulates; furthermore, the restrictions on the quantifiers in (5), (6), and (8) serve no useful purpose and can be eliminated. In particular, replacing (8) by the usual commutative law $\forall x \forall y[x \cdot y=y \cdot x]$, means that (12) can be eliminated. Finally, we observe that (7) can be replaced by $\forall x \exists y[x \neq 0 \rightarrow x \cdot y=1]$, since we have seen that $\forall x[x \cdot 0=0]$.

In this way, we obtain the usual postulate set for a field; hence, the following:

THEOREM. *The algebraic system* $(S, +, \cdot, 0, 1)$, *where* $+$ *and* $\cdot$ *are binary operators on* S, $0 \in S$ *and* $1 \in S$, *is a field if and only if each of the following propositions is true in the given algebraic system*:

(a) $\forall x \forall y \forall z[x+(y+z)=(x+y)+z]$

(b) $\forall x[x+0=x]$

(c) $\forall x \exists y[x+y=0]$

(d) $\forall x \forall y[x+y=y+x]$

(e) $\forall x \forall y \forall z[x \cdot(y \cdot z)=(x \cdot y) \cdot z]$

(f) $\forall x[x \cdot 1=x]$

(g) $\forall x \exists y[x \neq 0 \rightarrow x \cdot y=1]$

(h) $\forall x \forall y[x \cdot y=y \cdot x]$

(i) $\forall x \forall y \forall z[x \cdot(y+z)=x \cdot y+x \cdot z]$

(j) $0 \neq 1$.

IMAGINATIVE MATHEMATICS

NATHAN ALTSHILLER COURT, University of Oklahoma

I. Mathematics and Imagination.

a. *The Origin of numbers.* The title is ambiguous and, what is worse, presumptuous. Mathematics is cold, calculating, stiff. Such a forbidding thing could not have anything to do with the imagination.

Well, let's examine this question a little closer. Have you ever stopped to think where these things you compute with, these numbers, come from? Or better, suppose you were asked whether you remember the time when you could not count 1, 2, 3, 4, · · · . On a first impulse you would be ready to swear upon the thick beard of Pythagoras that there never was such a time. But your experience with young children makes you change your mind. No, numbers were not always with you.

The notion of number is an invention of man, a product of his imagination. Number is an abstraction man arrived at, slowly and painfully, by comparing groups or collections of objects. Thus the collection of eyes of an individual matches his collection of ears. Each of those two groups matches the collection of arms. None of those groups of objects matches the collection of fingers on the right hand or on the left hand. But the collection of fingers on one hand matches exactly the collection of fingers on the other, and those two collections put together match the collection of toes on both feet. And so on.

b. *Further Extensions.* This is the laborious way in which man mastered the notion of number and a hard won victory it was. But "ce n'est que le premier pas qui coûte" (only the first step is costly). Once on the march, man invented fractions, conquered rationally the irrational numbers, imagined the imaginary numbers, vectors, quaternions, tensors.

But those extensions of the number system apart, the integers alone provided for man enough opportunity to exercise his imagination. He created the enticing world of the theory of numbers, in which alongside with the well-known prime numbers and composite numbers, he considered perfect numbers, amicable numbers, deficient numbers, abundant numbers, etc. This theory also deals with such intriguing propositions as the number of prime numbers, the law of succession of prime numbers, Fermat's last theorem, etc., etc.

c. *Other Domains.* It is not necessary for me to elaborate the fact how much imagination is displayed in the practically inexhaustible geometrical elements and relations associated with as simple a thing as a triangle, that is, with three points, or with a circle. You are well familiar with this situation, and just reminding you of it will serve the purpose.

All that is just elementary mathematics. And what shall be said about the lofty, subtle, profound imaginings of higher mathematics, that are logically impeccable, but seem to stretch credibility almost to the breaking point?

II. Mathematics and Art.

a. *Mathematics and poetry.* You should, therefore, not be too surprised if I tell you that some people find a kinship between mathematics and the fine arts.

Many great and nearly great scholars, writers, and thinkers have spoken eloquently to this effect. Let me mention, to begin with, a New England Unitarian clergyman, Thomas Hill (1818–1891), the author of several books on mathematics. He said: "The Mathematics is usually considered the very antipodes of poesy. Yet mathesis and poesy are of the closest kindred, for they are both works of the imagination." Looking upon mathematics from another point of view, one of the most outstanding contemporary intellectuals, Bertrand Russell (born 1872) expressed the opinion: "The true spirit of delight, the exaltation, the sense of being more than a man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry."

b. *Mathematics as a fine art.* The American mathematicians Maxime Bôcher (1867–1918) and J. L. Coolidge (1873–1954) went further. They insisted that mathematics is itself a fine art. One of the leading English mathematicians of the 19th century, Jacob Joseph Sylvester (1814–1897) offered a graphical picture of the place mathematics occupies in the exalted company of the fine arts. He divides the whole domain of esthetics into Epics, Plastics, and Music, and adds to it mathematics as the fourth member of the noble group. He pictures the four members as the vertices of a tetrahedron.

Any three of the four branches of art have a common plane outside of which lies the fourth. Any two branches of art have a common axis, and the remaining two have their common axis, opposite the axis of the first two.

Moreover, any three members have a common center of gravity in their common plane. This point may be joined by a line to the vertex occupied by the fourth member of the art tetrad. The four lines thus obtained have a common center (Commandino's Theorem).

However, Sylvester confesses that what the common point may mean in the universe of esthetics he has not had time to figure out. The question thus remains an open challenge. Any volunteers?

c. *Art in science and science in art.* Let me add to this list one more quotation, this one from a contemporary Canadian biologist C. Leonard Huskins (born 1897): "I believe that there is more art in most sciences than is commonly recognized, and that different as the scientist and the artist may be, they are more like each other than either is like the members of any other group." (*American Scientist*, Vol. 39, No. 4, October 1951, p. 692.)

I would go Professor Huskins one better and say that the converse of his proposition is also true, namely, that there is more science in the arts than is commonly recognized.

I would even risk to stick out my neck and defend a much broader thesis. It seems to me that every science after it has reached a certain degree of development necessarily tends toward taking on characteristics of an art, and, conversely, every art after it has reached a certain stage has to incorporate a good deal of science and thus take on some characteristics of a science.

Let me try to give you an inkling of my idea. Any science begins by collecting a certain amount of empirical knowledge by observations. The time comes when it is necessary to organize and systematize that knowledge. Moreover, further knowledge is added to the stock by less direct methods, with the aid of reasoning, and simplicity becomes of essence. The farther the science progresses, the more it is compelled to make an effort towards order and elegance both in presenting the knowledge already accumulated and in the methods of discovery of new additions. If the process is pushed far enough, the situation becomes such that the adept of the science begins to feel that what he pursues is "more an art than a science."

On the other side we see that sculptors and painters must borrow from the geometer things like the "golden section" and particularly the theory of perspective. Art schools offer their students a course in "Anatomy for Artists." And so on.

III. Psychology in mathematics.

a. *The psychological element in a mathematical proof.* The cardinal requirement of a mathematical proof is that it should be logically sound; that is, that laws of logic should be used correctly. It is expected that the author himself should be convinced that he handles the logic properly, and that the other members of the mathematical fraternity who are competent to judge, should be of the opinion that the logic is valid. We thus come to the conclusion that the question of acceptability of a mathematical proposition is a matter of agreement of a group of people, or, in other words, is a matter of psychology.

b. *Authoritative Opinions.* The late Cassius Jackson Keyser (1862–1947), for many years chairman of the Department of Mathematics at Columbia University, has pointed out this fact in the following words: "Select a well-wrought demonstration and examine it. What can you say of it? You can say this: A normal human mind is such that if you begin with such-and-such principles or premises and with such-and-such ideas, and if you combine them in such-and-such an order, you will find that it passed from darkness to light, from doubt to conviction. Obviously such a proposition is not mathematical; it is psychological, it states a fact respecting the normal human mind."

Henri Lebesgue (1875–1941), of Lebesgue integral fame, expresses the same idea even more pointedly: "Les raisons de se déclarer satisfait par un raisonnement sont de nature psychologique en mathématiques, comme ailleurs" (The reasons for declaring oneself satisfied with a reasoning are of a psychological nature, in mathematics as in anything else.)

c. *Angle trisectors.* These remarks seem to shed a great deal of light on the baffling question of the inexhaustible supply of trisectors of angles and the squarers of the circle. If a man comes to you with a brand new and impeccable construction of the trisection of an angle, it does no good to tell him where he can find a proof that such a construction cannot be accomplished. Even if he is able to follow the argument in the article he is referred to, he is not convinced by the conclusion. He cannot find any fault with that argument, but he is sure that there is something wrong with it or with the result, or with both. Now why is it that you accept the proof as conclusive, and the angle trisector does not?

The reason is psychological, and there is more than one of such reasons for it.

In the first place, he has an emotional interest in his discovery and it is difficult for him to give it up. You do not have that handicap. What is more important, the propositions and methods, used in the proof that the trisection is impossible, bear a different weight with him from what they do with you. Those arguments were used by you before in other connections, and the results you obtained by them were correct, you have therefore full confidence in them. With your angle trisector, on the other hand, those same arguments are new, they seem to him to be invented *ad hoc*, just for the purpose of proving the impossibility of trisection. Instead of accepting the validity of the proof, your inventor would rather think that the mathematical fraternity was for such a long time unable to solve the problem that in self-protection they cooked up such a proof of its impossibility. But if you personally are willing to break the solid fraternal ranks and go over to the camp of the inventor, he will take you into partnership and share with you not only the fame but the enormous riches as well that he is bound to reap from the copyrighting of his epoch-making discovery. This may seem strange to you, but I am speaking from experience.

IV. The Principle of Workmanship. We could continue to discuss other implications of the psychological aspects of mathematical truths, but let us return to the relation between art and mathematics.

There are other aspects, besides imagination, in which mathematics vies with the arts. One of them may be called "workmanship." There may be some naive persons who imagine that the smooth and elegant poems they enjoy reading and rereading are the spontaneous products of the genius of the author. Better informed people know that rather the contrary is the case. The poet spends much time and effort before he obtains a version of his poem that he finds satisfactory. He disregards many a preliminary draft, works on every stanza, scrutinizes every line, weighs every word until the final product makes the impression of a natural, effortless, "inspired" creation.

A mathematician worthy of his name is just as ambitious, just as scrupulous about his work. His product must be well organized, his proofs logically faultless and at the same time as simple, as elegant as possible. If, carried away by the weight and value of his propositions, an author lets stand a proof that does not fulfill those requirements, someone else will come along and offer a better proof, then another mathematician may contribute an improvement of this newer proof. The results obtained this way are often surprising. Proofs that were originally based on advanced branches of mathematics may be reduced to a much lower level and in their simplicity may make the proposition itself nearly obvious. The mathematical profession is proud of the achievement.

Suppose you come across two triangles whose medians are respectively equal. Would that imply that the triangles are congruent? Euclid's classical method of superposition obviously does not work very readily. The proof would have to be more round about. The same would hold about the analogous problem involving the altitudes of the two triangles.

The latter problem was recently solved by Victor Thébault (1882–1960), a very gifted and prolific geometer whose name appeared frequently on the pages of the *American Mathematical Monthly* and this *Magazine.*

Let a, b, c; a', b', c', be the sides and h_a, h_b, h_c; h_a', h_b', h_c', the altitudes, and S, S' the areas of the two triangles. We have $2S/2S' = ah_a/a'h_a' = bh_b/b'h_b' = ch_c/c'h_c'$. Now if $h_a = h_a'$, $h_b = h_b'$, $h_c = h_c'$, then $a/a' = b/b' = c/c'$.

Thus the sides of the two triangles are proportional, and therefore the two triangles are similar. Now the ratio of any two corresponding linear elements of two similar triangles is the same as the ratio of similitude of the two triangles. In the present case the ratio of two corresponding altitudes is equal to unity, by assumption, hence this is the ratio of similitude of the triangles; that is, the triangles are congruent.

This is a very simple and very elegant proof. Yet it is not the simplest proof that can be devised. Suppose we took the three altitudes of one or the other of the two triangles and try to construct a triangle whose altitudes would be equal to those three given segments. Now this is a classical problem whose solution is known, and, moreover, the solution is unique; that is, one and only one such triangle can be constructed. From this construction the congruence of the two given triangles follows as an obvious corollary.

Why did Thébault not use this proof? He surely knew the construction problem involved. But I am not going to call Thébault a dumbbell, for he could have come back at me with the classical: "You are another." Why, that construction is fully displayed in my *College Geometry.* But that obvious corollary is not stated there, not even proposed as an exercise.

And this is not accidental. The problem of the congruence of two triangles with correspondingly equal medians is a corollary of an analogous construction problem which is solved in the same book. It is applicable to a considerable number of cases. For instance, two triangles are congruent if the radii of their respective escribed circles are equal.

Based on an address read before the National Council of Teachers of Mathematics at the twenty-second annual Summer meeting at the University of Wisconsin, Madison, August 17, 1962

ANTE UPMANSHIP

When gambling is suggested
I'm not enthusiastic.
I like my assets liquid
And strictly unstochastic.

The jump discontinuities
Of fortune are unceasing
And this becomes monotonous
When they are non-increasing.

MARLOW SHOLANDER

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MARLOW SHOLANDER

CONSECUTIVE INTEGERS WHOSE SUM OF SQUARES IS A PERFECT SQUARE

BROTHER U. ALFRED, F.S.C., St. Mary's College, California

The following investigation may be considered as one of the extensions of the familiar relation, $3^2+4^2=5^2$, in which from one point of view, two consecutive integers have the sum of their squares equal to a perfect square. The question is: May other sets of consecutive integers have the sum of their squares equal to a perfect square and if so, which?

Quite a few investigations of this matter have been made in the past and it has been found that if n is the number of consecutive integers then only for certain values of n is it possible to have the sum of the squares equal to a perfect square. Such values of n are 2, 11, 23, 24, 26, $\cdots$.

This article is primarily concerned with finding necessary conditions which n has to fulfill in order that there may be n successive integers the sum of whose squares is a perfect square. By setting up such necessary conditions, it will be possible to eliminate a great many values from further investigation. Of those that remain, additional study may produce further reductions in number. Finally, the values of n which are left after considering all available necessary conditions are studied so as to determine whether an actual instance of n consecutive integers whose sum is a perfect square can be found. This positive search for examples has been carried in the present instance to $n=500$.

Basic Equation and Basic Congruence. Let the n consecutive integers be $x, x+1, \cdots, x+n-1$ and let z be the square root of the sum of their squares. The equation to be satisfied is $x^2+(x+1)^2+(x+2)^2+ \cdots +(x+n-1)^2=z^2$ or

$$nx^2 + n(n-1)x + \frac{(n-1)n(2n-1)}{6} = z^2.$$

It is required to find integral values of x and z which satisfy this equation. The procedure to be followed may be illustrated by a simple example using the case $n=5$. Substituting into the above formula, we obtain

$$5x^2 + 20x + 30 = z^2.$$

Since 5 is a factor of all terms on the left, it must be a factor of z. Setting $z=5z'$, the equation becomes

$$x^2 + 4x + 6 = 5z'^2.$$

It now follows that 5 must be a factor of the left-hand side of the equation so that

$$x^2 + 4x + 6 \equiv 0 \pmod 5 \quad \text{or} \quad x^2 - x + 1 \equiv 0 \pmod 5).$$

Such a congruence determined from the basic equation shall be termed in this discussion the basic congruence. If it has no solution, then we know that for this value of n, we cannot have n consecutive integers the sum of whose squares is a perfect square. On the other hand, if there is a solution or solutions, this does not mean automatically that there will be n such numbers. The condition, in other words, is necessary, but it may not be sufficient. It may well be that on the basis of additional congruence considerations, an impasse will be reached. For example, in the case of $n=13$, we arrive at a basic congruence

$$x^2 - x - 2 \equiv 0 \pmod{13}$$

which has two solutions, $x \equiv 2, -1 \pmod{13}$. Now the basic equation for $n=13$ is

$$13x^2 + 156x + 650 = z^2$$

with every term on the left divisible by 13. After setting $z=13z'$, this becomes

$$x^2 + 12x + 50 = 13z'^2.$$

Using the first solution, $x \equiv 2 \pmod{13}$, we set $x=13x'+2$ and substitute. This results in the equation:

$$13x'^2 + 16x' + 6 = z'^2.$$

If $x' \equiv 1 \pmod 2$, we find that $z'^2 \equiv 3 \pmod 4$ from this equation; if $x' \equiv 0 \pmod 2$, it follows that $z'^2 \equiv 2 \pmod 4$. But since every square must be congruent to either 0 or 1 modulo 4, it is apparent that the above equation cannot have a solution. We arrive at a similar negative result by using the other value $x \equiv -1 \pmod{13}$.

Values of n congruent to 4, 5, or 6 modulo 8. Before taking up the main line of development, two special cases will be dealt with. The first concerns values of n of the forms $8r+4$, $8r+5$, and $8r+6$. If there are four consecutive numbers, two must be odd and two even. The square of an even number being congruent to 0 modulo 4 and the square of an odd number being congruent to 1 modulo 4, the sum of the squares of four consecutive integers must be congruent to 2 modulo 4. Hence such a sum cannot be a square. Moreover, if we add any multiple of 8 to 4, we arrive at the same condition. Thus all values of n of the form $8r+4$ are excluded.

For $n=5$, there could be two odd numbers and three even numbers or three odd numbers and two even numbers. In the former case, the sum of the squares is congruent to 2 modulo 4; in the latter, to 3 modulo 4. Hence no value of n of the form $8r+5$ has a solution.

In the case $n=6$, three numbers are odd and three even, again leading to the conclusion that the sum of the squares cannot be a perfect square. Hence, initially in examining a series of values of n all those of forms $8r+4$, $8r+5$, and $8r+6$ can be excluded from consideration.

n a perfect Square $\not\equiv 0$ modulo 2 or 3. If n is the square of an odd number not divisible by 3, it can be shown in general that there is always a solution except for the case of 5^2. Without giving the complete formulation, we may illustrate the process for the particular instance $n=169$. The basic equation is

$$169x^2 + 169\cdot 168x + 169\cdot 9436 = z^2.$$

Setting $z=13z'$, this becomes $x^2+168x+9436=z'^2$.

We now form a perfect square and obtain: $(x+84)^2+2380=z'^2$. There remains simply to determine in what ways 2380 can be represented as the sum of consecutive odd integers. This problem can be solved in a general fashion so that we may conclude the existence of a finite number of solutions with a straightforward method of determining them. In the present instance, two consecutive odd integers 1189 and 1191 add up to 2380. This means that $x+84 =(1189-1)/2=594$ and hence that $x=510$, $z'=594+2=596$, and $z=13z' =7748$.

We can check this result as follows:

$$\sum_{k=510}^{678} k^2 = \frac{678\times 679\times 1357}{6} - \frac{509\times 510\times 1019}{6}$$
$$= 104{,}118{,}539 - 44{,}087{,}035 = 60{,}031{,}504.$$

This agrees with the square of 7748.

A second solution is found: the sum of the squares of the consecutive integers from 30 to 198 inclusive is the square of 1612.

Notation and Classification of Cases. In the more general development that follows, we shall be dealing with many different cases so that it is advisable to set down at this point additional notation over and above what has already been given. For our purposes, it will be convenient to represent n as

$$n = 2^\alpha 3^\beta q^2 n',$$

where α and $\beta \geqq 0$, $q^2 \not\equiv 0$ modulo 2 or 3, and $n' \not\equiv 0$ modulo 2 or 3 with no prime factor of power greater than unity. The series of classes that will be studied are the following:

(1) n of the form n',

(2) n of the form q^2n',

(3) n of the form $2^\alpha 3^\beta$,

(4) n of the form $2^\alpha 3^\beta q^2$,

(5) n of the form $2^\alpha 3^\beta n'$,

(6) n of the form $2^\alpha 3^\beta q^2 n'$.

CASE 1: *n of the form n'.* In the investigation of specific values of n, it was noted empirically that there were no solutions of the basic congruence for quantities of the form $12\lambda+5$ or $12\lambda+7$, whereas for those of the form $12\lambda+1$ or $12\lambda-1$, there might or might not be a solution. As a result, it was decided to seek an explanation of this phenomenon.

First of all, the basic congruences for numbers of these types were worked out. This process will be illustrated for the case $n=12\lambda+5$. The basic equation is

$$(12\lambda+5)x^2 + (12\lambda+5)(12\lambda+4)x + \frac{(12\lambda+4)(12\lambda+5)(24\lambda+9)}{6} = z^2.$$

Since $12\lambda+5$ is not divisible by 2 or 3 and since it has no prime factors of degree greater than unity, $12\lambda+5$ must be a factor of z. We set $z=(12\lambda+5)z'$. Substitution gives the equation

$$x^2 + (12\lambda + 4)x + (6\lambda + 2)(8\lambda + 3) = (12\lambda + 5)z'^2$$

which leads to the basic congruence

$$x^2 - x \equiv -(2\lambda + 1) \pmod{12\lambda + 5}.$$

For the other cases of n',

$$\begin{aligned} x^2 - x &\equiv 2\lambda && \pmod{12\lambda + 1}, \\ x^2 - x &\equiv 2\lambda + 1 && \pmod{12\lambda + 7}, \\ x^2 - x &\equiv -2\lambda && \pmod{12\lambda - 1}, \end{aligned}$$

n' a Prime Number.

A. $n'=12\lambda+1$. The basic congruence can be transformed into

$$4x^2 - 4x + 1 = (2x - 1)^2 \equiv 8\lambda + 1 \pmod{12\lambda + 1}.$$

If there is to be a solution, $8\lambda+1$ must be a quadratic residue of $12\lambda+1$. To determine whether this is the case we employ the Jacobian and the quadratic reciprocity law. (See Uspensky and Heaslett, *Elementary Number Theory*, McGraw-Hill, 1939, pp. 295 ff.) We have to evaluate

$$\left(\frac{8\lambda + 1}{12\lambda + 1}\right).$$

Since the denominator is prime by supposition it follows that if this quantity is $+1$, then $8\lambda+1$ is a quadratic residue of $12\lambda+1$ and there is a solution; if the value is -1, it is not a quadratic residue and hence there is no solution.

The evaluation is as follows:

$$\left(\frac{8\lambda + 1}{12\lambda + 1}\right) = \left(\frac{12\lambda + 1}{8\lambda + 1}\right) = \left(\frac{4\lambda}{8\lambda + 1}\right) = \left(\frac{\lambda}{8\lambda + 1}\right)$$
$$= \left(\frac{8\lambda + 1}{\lambda}\right) = \left(\frac{1}{\lambda}\right) = +1.$$

Therefore, for a prime of the form $12\lambda+1$, there is always a solution of the basic congruence.

B. *n' a Prime* $12\lambda+5$. As before, we transform the original congruence so as to have a perfect square on the left-hand side:

$$(2x - 1)^2 \equiv -8\lambda - 3 \equiv 4\lambda + 2 \pmod{12\lambda + 5}.$$

We evaluate the Jacobian as follows:

$$\left(\frac{4\lambda + 2}{12\lambda + 5}\right) = \left(\frac{2}{12\lambda + 5}\right)\left(\frac{2\lambda + 1}{12\lambda + 5}\right).$$

Two cases may be distinguished: (1) $\lambda=2\lambda'$; (2) $\lambda=2\lambda'+1$.

(1) $\lambda = 2\lambda'$

$$\left(\frac{4\lambda+2}{12\lambda+5}\right)=\left(\frac{2}{24\lambda'+5}\right)\left(\frac{4\lambda'+1}{24\lambda'+5}\right)=-\left(\frac{24\lambda'+5}{4\lambda'+1}\right)$$
$$=-\left(\frac{-1}{4\lambda'+1}\right)=-1.$$

(2) $\lambda = 2\lambda'+1$

$$\left(\frac{4\lambda+2}{12\lambda+5}\right)=\left(\frac{2}{24\lambda'+17}\right)\left(\frac{4\lambda'+3}{24\lambda'+17}\right)=\left(\frac{24\lambda'+17}{4\lambda'+3}\right)$$
$$=\left(\frac{-1}{4\lambda'+3}\right)=-1.$$

Hence in the case of a prime of the form $12\lambda+5$, there is no solution. By means of similar proofs, it is found that for a prime $12\lambda+7$, no solution exists, while for a prime $12\lambda-1$, there is always a solution of the basic congruence.

n' a Product of Two Unequal Primes. Empirically, a very curious fact was noted in the case of composite moduli of the form $12\lambda+1$, $12\lambda+5$, $12\lambda+7$, and $12\lambda-1$. This may be illustrated for $n=1001=11\cdot 13\cdot 7=12\cdot 83+5$.

The basic congruence being $x^2-x\equiv-(2\lambda+1)(\text{mod } 12\lambda+5)$ we have $x^2-x\equiv-167$ (mod 1001). For this to have a solution, each of the equations formed with the factors as moduli must have solutions. These subsidiary congruences are

$$x^2-x\equiv-167 \pmod 7\equiv 1 \pmod 7$$
$$x^2-x\equiv-167 \pmod{11}\equiv-2 \pmod{11}$$
$$x^2-x\equiv-167 \pmod{13}\equiv 2 \pmod{13}.$$

Now the curious fact is that these are precisely the basic congruences for each of the three prime moduli! It is possible to prove that such is always the case. The method of doing so will be illustrated for a two-factor composite of the form $12\lambda+7$.

The factors of such a number might be either the pair $12\lambda+7$ and $12\lambda+1$ or the pair $12\lambda+5$ and $12\lambda-1$ in form.

$$12\lambda+7=(12\theta_1+7)(12\theta_2+1)=144\theta_1\theta_2+12\theta_1+84\theta_2+7$$
$$(1)\qquad \lambda=12\theta_1\theta_2+\theta_1+7\theta_2$$
$$2\lambda+1=24\theta_1\theta_2+2\theta_1+14\theta_2+1.$$

Hence $x^2-x\equiv 2\theta_1+1$ (mod $12\theta_1+7$) and $x^2-x\equiv 14\theta_2+1\equiv 2\theta_2$ (mod $12\theta_2+1$) which shows that the congruences for the individual factors are the same as given by the formulas for the basic congruences of these numbers.

$$12\lambda + 7 = (12\theta_1 + 5)(12\theta_2 - 1) = 144\theta_1\theta_2 - 12\theta_1 + 60\theta_2 - 5$$

$$\lambda = 12\theta_1\theta_2 - \theta_1 + 5\theta_2 - 1 \tag{2}$$

$$2\lambda + 1 = 24\theta_1\theta_2 - 2\theta_1 + 10\theta_2 - 1.$$

Hence

$$x^2 - x \equiv -(2\theta_1 + 1) \pmod{12\,\theta_1 + 5}$$

and

$$x^2 - x \equiv 10\theta_2 - 1 \equiv -2\theta_2 \pmod{12\theta_2 - 1}.$$

Again we arrive at the basic congruences of the individual factors. For all cases of n' consisting of two factors, we find that the existence of a solution of the basic congruence depends on whether there are solutions for both factors. This means that neither can be of the form $12\lambda+5$ or $12\lambda+7$. It might be pointed out, before arriving at our general conclusion, that it is easy to see that we can go from two factors to three, from three to four, and so on indefinitely by successive groupings. Also, any composite n' of the form $12\lambda+5$ or $12\lambda+7$ must have at least one factor of the form $12\lambda+5$ or $12\lambda+7$. On the other hand, composites n' of the form $12\lambda+1$ or $12\lambda-1$ may or may not have such factors. Hence we arrive at the following conclusions:

For values of n equal to n', a quantity not congruent to zero modulo 2 or 3 and having no factor of degree greater than 1, there will be no solution of the basic congruence for all numbers of the form $12\lambda+5$ or $12\lambda+7$; for numbers n' of the form $12\lambda+1$ or $12\lambda-1$, there will be no solution if any factor is of the form $12\lambda+5$ or $12\lambda+7$; otherwise, there will be a solution.

CASE 2: *n of the form q^2n'*. If n contains certain square factors not congruent to zero modulo 2 or 3, it can be shown that these may be eliminated and that we arrive at the same congruence for n' as before. The proof for $n'=12\lambda+1$ will be given to show the method of procedure.

A partial substitution of $n=q^2\,n'$ into the basic equation

$$nx^2 + n(n-1)x + \frac{(n-1)n(2n-1)}{6} = z^2$$

gives

$$q^2n'x^2 + q^2n'(n-1)x + \frac{(n-1)q^2n'(2n-1)}{6} = z^2.$$

Since both q^2 and n' are prime to 2 and 3, we can set $z=qn'z'$. On substituting the equation becomes:

$$x^2 + (n-1)x + \frac{(n-1)(2n-1)}{6} = n'z'^2.$$

We note that q^2 is of the form $12\rho+1$ since q is either $12\phi\pm 1$ or $12\phi\pm 5$ in form.

So for the case of $n'=12\lambda+1$,

$$n = q^2 n' = (12\rho + 1)(12\lambda + 1)$$
$$n - 1 = 144\rho\lambda + 12\rho + 12\lambda$$
$$\frac{(n-1)(2n-1)}{6} = (24\rho\lambda + 2\rho + 2\lambda)(2n - 1) \equiv -2\lambda \pmod{12\lambda + 1}.$$

Hence $x^2-x\equiv 2\lambda \pmod{12\lambda+1}$. Similar results follow for the other forms of n'. Hence we can formulate the following more general conclusion:

Any value of n not congruent to zero modulo 2 or 3 will not have a solution of the basic congruence if it contains an odd power of a prime factor of the form $12\lambda+5$ or $12\lambda+7$; otherwise, it will have a solution.

Special Result. Before proceeding to Case (3), an intermediate consideration applying to all values of n will be taken up. If n is of the form $2^{2\alpha}m$, where $m\not\equiv 0 \pmod 2$, the basic equation becomes

$$2^{2\alpha}mx^2 + 2^{2\alpha}m(n-1) + \frac{2^{2\alpha-1}m(n-1)(2n-1)}{3} = z^2.$$

Let $z=2^\alpha z'$. Then

$$2mx^2 + 2m(n-1)x + \frac{m(n-1)(2n-1)}{3} = 2z'^2.$$

Since all terms except the constant term are divisible by 2, it follows that this equation cannot have an integral solution. Hence, if n contains as factor an even power of 2, no solution is possible. A similar result follows if n has an even power of 3 as factor.

CASE 3: *n of the form $2^\alpha 3^\beta$.* The results in this case may be summarized as follows:

(1) For $n=2$, there are solutions.
(2) For $n=2^\alpha$; $\alpha>1$, there is no solution.
(3) There are no solutions if $n=3^\beta$.
(4) No solution exists for n of the form $2\cdot 3^{2\beta+1}$.
(5) A solution may exist for n of the form $2^{2\alpha+1}3^{2\beta+1}$ with $\alpha\geqq 1$ and $\beta\geqq 0$.

None of these proofs will be given as there is nothing particularly distinctive about them and the type of approach will be seen in greater complexity for other cases.

CASE 4: *n of the form $2^\alpha 3^\beta q^2$.* The summary of results for this case is as follows:

(1) For $n=q^2$, there is always a solution if $q>5$.
(2) For $n=2q^2$, solutions are possible. Examples are $n=50$ and 242.
(3) No solutions are possible for $n=3q^2$.
(4) No solutions are possible for $n=6q^2$.

(5) For $n=2^{2\alpha+1}q^2$, $\alpha \geqq 1$, no solution is possible.
(6) For $n=3^{2\beta+1}q^2$, no solution is possible.
(7) For $n=2\cdot 3^{2\beta+1}q^2$, $\beta \geqq 1$, solutions are possible.
(8) For $n=2^{2\alpha+1}3^{2\beta+1}q^2$, $\alpha>1$, $\beta\geqq 0$, solutions are possible.

Again, we pass over the proofs.

CASE 5: *n of the form* $2^\alpha 3^\beta n'$. We consider only odd powers of 2 and 3 inasmuch as we have already proved that even powers lead to no solution. With this limitation, it can be shown that in all instances, we arrive at the same congruence as that for n' and hence the existence or non-existence of a solution depends on the nature of n' (see Case 1). The proofs involved developed a certain measure of complexity. Three sub-cases were considered:

(A) $2^{2\alpha+1}n'$; (B) $3^{2\beta+1}n'$; (C) $2^{2\alpha+1}3^{2\beta+1}n'$.

Under each, n' was taken successively in the forms $12\lambda+1$, $12\lambda+5$, $12\lambda+7$, and $12\lambda-1$. Typical of the work involved is the development for $n=2^{2\alpha+1}n'$, where n' is of the form $12\lambda+7$.

Taking $n=2^{2\alpha+1}(12\lambda+7)$ and making a partial substitution into the basic equation, we obtain

$$2^{2\alpha+1}n'x^2+2^{2\alpha+1}n'(n-1)x+\frac{(n-1)2^{2\alpha}n'(2n-1)}{3}=z^2.$$

Let $z=2^\alpha n'z'$. Then

$$2x^2+2(n-1)x+\frac{(n-1)(2n-1)}{3}=n'z'^2.$$

The problem reduces to evaluating the constant term in the equation and finding its congruent value modulo $n'=12\lambda+7$. More specifically, we need only evaluate $(2n-1)/3$.

$$2n-1=2^{2\alpha+2}\cdot 12\lambda+7\cdot 2^{2\alpha+2}-1$$

$$\frac{2n-1}{3}=4\lambda\cdot 2^{2\alpha+2}+2^{2\alpha+3}+\frac{(2^2-1)}{3}(2^{2\alpha}+2^{2\alpha-2}+\cdots+2^2+1)$$

$$=\frac{(2^{2\alpha+2}-1)}{3}\cdot 12\lambda+4\lambda+2^{2\alpha+3}+(2^{2\alpha}+2^{2\alpha-2}+\cdots+2^2+1).$$

Using the fact that $7=2^2+(2^2-1)$, we can put this into the form

$$\frac{2n-1}{3}=(2^{2\alpha}+2^{2\alpha-2}+\cdots+2^2+1)(12\lambda+7)+4\lambda+2.$$

Hence

$$\frac{(n-1)(2n-1)}{3}\equiv-(4\lambda+2)\pmod{12\lambda+7}$$

and $2x^2-2x-(4\lambda+2)\equiv 0 \pmod{12\lambda+7}$ or $x^2-x\equiv 2\lambda+1 \pmod{12\lambda+7}$ which is the form of the basic congruence for $n'=12\lambda+7$.

CASE 6: *n of the form* $2^{2\alpha+1}3^{2\beta+1}q^2n'$. It was necessary to consider the following cases in covering the different situations under this heading: (1) $2q^2n'$; (2) $3q^2n'$; (3) $6q^2n'$; (4) $2^{2\alpha+1}q^2n'$; (5) $3^{2\beta+1}q^2n'$; (6) $2\cdot 3^{2\beta+1}q^2n'$; (7) $2^{2\alpha+1}3^{2\beta+1}q^2n'$. As typical of the type of proof involved, let us consider the case $n=2^{2\alpha+1}q^2n'$ with n' of the form $12\lambda+1$. The basic equation becomes on partial substitution

$$2^{2\alpha+1}q^2n'x^2 + 2^{2\alpha+1}q^2n'(n-1)x + \frac{2^{2\alpha}q^2n'(n-1)(2n-1)}{3} = z^2.$$

Let $z=2^{\alpha}qn'z'$. Then

$$2x^2 + 2(n-1)x + \frac{(n-1)(2n-1)}{3} = n'z'^2.$$

Now q^2 is of the form $3\phi+1$. Hence:

$$\begin{aligned}\frac{(n-1)(2n-1)}{3} &= (n-1)\left[\frac{2^{2\alpha+2}(3\phi+1)(12\lambda+1)-1}{3}\right]\\ &\equiv -\left[\frac{2^{2\alpha+2}12\lambda+2^{2\alpha+2}-1}{3}\right] \pmod{12\lambda+1}\\ &\equiv -\left[(12\lambda+1)\left(\frac{2^{2\alpha+2}-1}{3}\right)+4\lambda\right] \pmod{12\lambda+1}\\ &\equiv -4\lambda \pmod{12\lambda+1}.\end{aligned}$$

Therefore $2x^2-2x-4\lambda\equiv 0 \pmod{12\lambda+1}$ or $x^2-x\equiv 2\lambda \pmod{12\lambda+1}$.

The general conclusion from all the investigations in this case is quite simple: in all instances, the basic congruence reduces to that for n'.

This concludes the account of the development of the necessary conditions that must be fulfilled by n. While the investigation was replete with many divisions and distinctions, it turns out that the final summary is relatively simple.

Summary of excluded values of n.

(1) No integer of the form $8\lambda+4$, $8\lambda+5$, or $8\lambda+6$.
(2) No integer of the form $12\lambda+5$ or $12\lambda+7$.
(3) No number containing as factor an odd power of a prime of the form $12\lambda+5$ or $12\lambda+7$.
(4) No number containing as factor an even power of 2 or an even power of 3.
(5) All numbers with no other factors but 2 or 3 with the exception of 2 and $2^{2\alpha+1}3^{2\beta+1}$ with $\alpha\geqq 1$.
(6) All numbers of the form $2^{2\alpha+1}3^{2\beta+1}q^2$ with the exception of $q^2(q>5)$, $2q^2$, $2\cdot 3^{2\beta+1}q^2(\beta\geqq 1)$, and $2^{2\alpha+1}3^{2\beta+1}q^2(\alpha\geqq 1, \beta\geqq 0)$.

Application of the necessary conditions to a series of integers. The effectiveness of these necessary conditions in determining what values n may take can be brought out quite practically by considering the 50 numbers from 201 to 250. We shall list the number and the reason it is rejected in case there is no solution.

n	*Solution?*	*Reason*
201	no	$3 \cdot 67$. Factor $12\lambda+7$.
202	no	$2 \cdot 101$. Factor $12\lambda+5$.
203	no	$7 \cdot 29$. Factors $12\lambda+7$, $12\lambda+5$.
204	no	$8\lambda+4$.
205	no	$8\lambda+5$.
206	no	$8\lambda+6$.
207	no	$3^2 \cdot 23$. Even power of 3.
208	no	$2^4 \cdot 13$. Even power of 2.
209	no	$12\lambda+5$.
210	no	$2 \cdot 3 \cdot 5 \cdot 7$. Factors $12\lambda+5$, $12\lambda+7$.
211	no	$12\lambda+7$.
212	no	$8\lambda+4$.
213	no	$8\lambda+5$.
214	no	$8\lambda+6$.
215	no	$5 \cdot 43$. Factors $12\lambda+5$, $12\lambda+7$.
216		
217	no	$7 \cdot 31$. Factors $12\lambda+7$.
218		
219		
220	no	$8\lambda+4$.
221	no	$8\lambda+5$.
222	no	$8\lambda+6$.
223	no	$12\lambda+7$.
224	no	$2^5 \cdot 7$. Factor $12\lambda+7$.
225	no	$3^2 5^2$. Even power of 3.
226	no	$2 \cdot 113$. Factor $12\lambda+5$.
227		
228	no	$8\lambda+4$.
229	no	$8\lambda+5$.
230	no	$8\lambda+6$.
231	no	$3 \cdot 7 \cdot 11$. Factor $12\lambda+7$.
232	no	$2^3 \cdot 29$. Factor $12\lambda+5$.
233	no	$12\lambda+5$.
234	no	$2 \cdot 3^2 \cdot 23$. Even power of 3.
235	no	$12\lambda+7$.
236	no	$8\lambda+4$.
237	no	$8\lambda+5$.
238	no	$8\lambda+6$.
239		
240	no	$2^4 \cdot 3 \cdot 5$. Even power of 2.
241		
242		
243	no	3^5. Power of 3.
244	no	$8\lambda+4$.
245	no	$8\lambda+5$.
246	no	$8\lambda+6$.
247	no	$12\lambda+7$.
248	no	$2^3 \cdot 31$. Factor $12\lambda+7$.
249		
250	no	$2 \cdot 5^3$. Odd power of factor $12\lambda+5$.

The next step in the process is to consider those numbers which are not excluded on the basis of the necessary conditions for satisfying the basic congruence. In some instances, as was pointed out at the beginning of this paper, it is possible to eliminate the value of n by means of additional considerations. The pattern is not uniform and one must exercise a certain amount of ingenuity, but in general the idea is to take the basic equation or some derived equation and test it modulo various numbers.

As an example, the case of 216 may be introduced. The basic equation is

$$216x^2 + 216\cdot 215x + \frac{215\cdot 216\cdot 431}{6} = z^2.$$

After dividing by 6, we find that there is a factor of 36 in each term of the left-hand side of the equation. Hence we can set $z=6z'$, a substitution that leads to

$$6x^2 + 1290x + 92665 = z'^2.$$

Taking this equation modulo 7, we have

$$-x^2 + 2x - 1 \equiv z'^2 \pmod 7$$

for $x\equiv 0$, $z'^2\equiv 6$; $x\equiv 1$, $z'^2\equiv 0$; $x\equiv 2$, $z'^2\equiv 6$; $x\equiv 3$, $z'^2\equiv 3$; $x\equiv -1$, $z'^2\equiv 3$; $x\equiv -2$, $z'^2\equiv 5$; $x\equiv -3$, $z'^2\equiv 5$; all taken modulo 7. Now the quadratic residues of 7 are 0, 1, 2, and 4. Hence $x\equiv 1$ and $z'\equiv 0$ modulo 7 is the only possible set of values. Let $x=7x'+1$ and $z'=7z''$. Then

$$42x'^2 + 1302x' + 13423 = 7z''^2.$$

All coefficients are divisible by 7 with the exception of the constant term 13423. Hence there is no solution.

Determining solutions. The method of seeking solutions will be illustrated for the case of $n=431$. The basic equation is:

$$431x^2 + 431\cdot 430x + \frac{430\cdot 431\cdot 861}{6} = z^2.$$

Let $z=431z'$. Then $x^2+430x+61705=431z'^2$. Hence $x^2-x+72\equiv 0 \pmod{431}$ and $x\equiv 210, -209 \pmod{431}$.

Taking the first solution, $x=431x'+210$ which on substitution into the equation $x^2+430x+61705=431z'^2$ gives

$$431x'^2 + 850x' + 455 = z'^2.$$

If x' is even, $z'^2\equiv 3 \pmod 4$; if x' is odd, $z'^2\equiv 0 \pmod 4$. So x' must be odd. Set $x'=2x''+1$, $z'=2z''$. Then

$$431x''^2 + 856x'' + 434 = z''^2.$$

Again, x'' must be odd. Substitute $x''=2y+1$. We obtain

$$1724y^2 + 3436y^2 + 1721 = z''^2.$$

Let Δ be the change in the value of z''^2 as y changes by 1. Then $\Delta = 3448y + 5160$.

The value of z''^2 for $y=1$ is 6881; the value of Δ for $y=1$ is 8608. The procedure was to set 6881 into the calculator, then add 8608 to get the value of z''^2 for $y=2$; then 8608 is changed by 3448 and the sum added to the value of z''^2 for $y=2$ to get the value of z''^2 for $y=3$; and so on. If any of the resulting values of z''^2 had terminating digits that indicated it might be a square, the number was looked up in Barlow's Tables of Squares.

In the case of 431, the answer came very quickly for $y=5$. This gave $z''^2 = 62{,}001 = 249^2$. Then $z' = 2z'' = 498$; $z = 431z' = 214638$. $x'' = 2y+1 = 11$; $x' = 2x'' + 1 = 23$; $x = 431x' + 210 = 10123$.

We calculate

$$\sum_{10123}^{10563} k^2$$

which is found to be 46,069,471,044 in agreement with the square of 214,638.

Summary of numerical results to $n=500$. In the following table the results for all values not excluded by necessary conditions are summarized. In the first column, is n, the number of consecutive integers; in the second, x, the first of the integers; in the third, $x+n-1$, the last; in the fourth, z, the square root of the sum of the squares of the n integers.

It will be noted that there is no solution for a number of values of n. The reason is that it was not possible to exclude them by any necessary conditions, on the one hand, and on the other, no solution was obtained within the limits of the calculations.

TABLE: *Showing Consecutive Integers Whose Sum of Squares is a Perfect Square.*

n	x	$x+n-1$	z
2	3	4	5
2	20	21	29
2	119	120	169
11	18	28	77
11	38	48	143
11	456	466	1,529
11	854	864	2,849
11	9,192	9,202	30,503
11	17,132	17,142	56,837
23	7	29	92
23	17	39	138
23	881	903	4,278
23	1,351	1,373	6,532
24	1	24	70
24	9	32	106
24	20	43	158
24	25	48	182
24	44	67	274
24	76	99	430
24	121	144	650

TABLE: *Showing Consecutive Integers Whose Sum of Squares is a Perfect Square.* (*Continued*)

n	x	$x+n-1$	z
24	197	220	1,022
24	304	327	1,546
24	353	376	1,786
24	540	563	2,702
24	856	879	4,250
24	1,301	1,324	6,430
24	2,053	2,076	10,114
24	3,112	3,135	15,302
26	301	326	1,599
33	27	59	253
47	731	777	5,170
49	25	73	357
50	7	56	245
59	22	80	413
59	1,438	1,496	11,269
73	442	514	4,088
74	3,185	3,258	27,713
88	192	279	2,222
88	225	312	2,530
96	13	108	652
97	15	111	679
107			
121	244	364	3,366
122	50	171	1,281
146	6,567	6,712	80,227
169	510	678	7,748
169	30	198	1,612
177	553	729	8,555
184	7	190	1,518
191	4,493	4,683	63,412
193			
194	112	305	3,007
218	590	807	10,355
227			
239	775	1,013	13,862
241	3,807	4,047	60,973
242	64	305	3,069
249	556	804	10,790
275			
289	20	308	3,128
289	140	428	5,032
289	199	487	6,001
289	287	575	7,463
289	433	721	9,911
289	724	1,012	14,824
289	1,595	1,883	29,597
297	170	466	5,676
299	132	430	5,083
299	168	466	5,681
311	2,277	2,587	42,918
312	15	326	3,406
313	1,788	2,100	34,430

TABLE: *Showing Consecutive Integers Whose Sum of Squares is a Perfect Square.* (*Continued*)

n	x	$x+n-1$	z
337	5,063	5,399	96,045
338	7,672	8,009	144,157
347	11,320	11,666	214,099
352	280	631	8,756
361	358	718	10,412
361	722	1,082	17,252
361	2,534	2,894	51,604
362	1,805	2,166	37,829
376	966	1,341	22,466
383	11,081	11,463	220,608
393	5,618	6,010	115,280
407	8,915	9,321	183,964
409	71,752	72,160	1,455,222
431	10,123	10,553	214,638
443	16,806	17,248	358,387
457			
458	1,081	1,538	28,167
479	7,989	8,467	180,104
481	45,619	46,099	1,005,771
491	12,584	13,074	284,289

Problems for solution. The following are some of the questions that remain for solution:

(1) A resolution of the impasse mentioned above for the particular values of n: 107; 193; 227; 275; 457.

(2) Determining whether the number of solutions is finite or infinite given that there is at least one solution. In the case of $n=2$, we have the following recursion formulas:

$$z_\lambda = 6z_{\lambda-1} - z_{\lambda-2} \quad \text{and} \quad x_\lambda = 6x_{\lambda-1} - x_{\lambda-2} + 2(z_0 = 1,\ x_0 = 0)$$

for obtaining an infinite set of solutions. On the other hand, we have found that for n equal to perfect squares greater than 5^2 and not congruent to zero modulo 2 or 3, the number of solutions is finite. What, then, is the general situation?

(3) Determining as for the case of $n=2$ a recursion formula that would give the infinite set of solutions should such exist. Considerable work was done for the case $n=24$ and while certain regularities were noted, it was not possible to arrive at a successful conclusion.

CORRECTION TO "FACTORIZATION OF INTEGERS"

WILLIAM EDWARD CHRISTILLES, St. Mary's University, Texas

In Christilles' paper, "Factorization of Integers," Vol. 36 (1963), the second sentence in the third paragraph, page 34, should have read:

"Although the restriction that the discriminant, D, be positive and not the square of an integer guarantees that none of the a_i or c_i be zero when dealing with reduced forms, some of the α_i or γ_i may be zero."

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BOOLEAN MATRICES AND LOGIC

F. D. PARKER, University of Alaska (now at SUNY at Buffalo)

Introduction. In a mathematical system, if the postulates are of the form A implies B $(A \rightarrow B)$, it is possible to prove theorems in the system by displaying the postulates as a Boolean matrix and examining the powers of that matrix. In fact, it is possible in a limited sense to discover all the theorems in the system. Moreover, it is also possible to use the matrices to determine whether two systems of postulates using the same language are equivalent, and some of the dependent postulates can be eliminated. It will come as no surprise to the electrical engineer who uses these matrices to analyze switching circuits that simple systems can be built electrically and the theorems discovered by watching a panel of lights.

Simple Mathematical Systems. Given a set of undefined terms and a set of conditions $A_1, A_2, A_3, \cdots, A_n$, together with a set of postulates of the form $A_i \rightarrow A_j$, it is possible to construct a diagram with appropriate arrows and de-

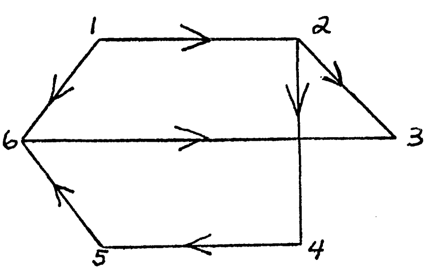

FIG. 1

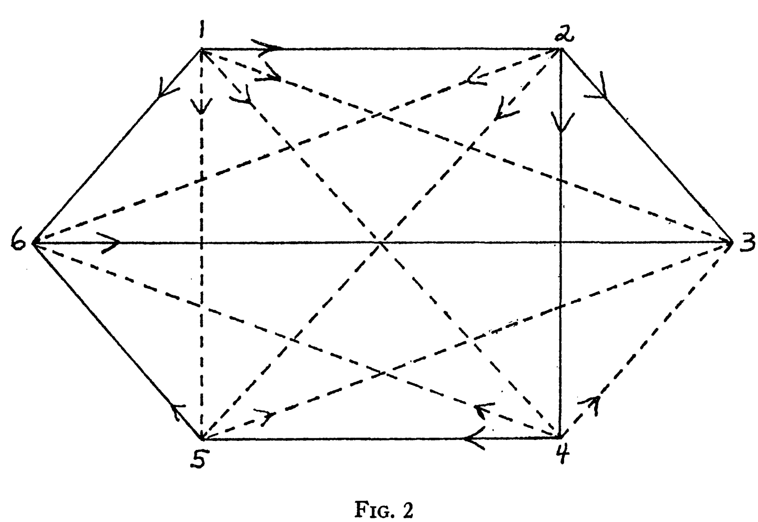

FIG. 2

termine whether or not $A_p \rightarrow A_q$. For example, suppose we have $A_1 \rightarrow A_2$, $A_2 \rightarrow A_3$, $A_2 \rightarrow A_4$, $A_4 \rightarrow A_5$, $A_6 \rightarrow A_3$, $A_1 \rightarrow A_6$, $A_5 \rightarrow A_6$.

Figure 1 shows these postulates. We can see from the figure that $A_2 \rightarrow A_6$, but $A_6 \nrightarrow A_2$. It is possible by tracing the directed arrows to test any proposition, and eventually to find all propositions of the form $A_i \rightarrow A_j$. We can diagram each proposition with dotted lines until these propositions are all discovered, when this system now yields Figure 2. If a larger and more complicated system of postulates exists, the work may be somewhat laborious, but fortunately there exists another method. We introduce an $n \times n$ matrix with element 1 in the ith row, jth column if $A_i \rightarrow A_j$, otherwise 0 and consider the elements as Boolean elements, subject to the laws $0 \cdot 0 = 1 \cdot 0 = 1 \cdot 0 = 0$, $1 \cdot 1 = 1$, $1+1=0+1=1+0=1$, $0+0=0$. Since $A_i \rightarrow A_i$, the matrix will be reflexive. The matrix describing Figure 1 is

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Matrix multiplication shows that

$$M^2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

The unit element now represents either a postulate or a proposition proved in two steps (i.e. $A_1 \rightarrow A_2$, $A_2 \rightarrow A_3$, $\therefore A_1 \rightarrow A_3$), which explains the unit in position (13). The propositions which are either postulates, two-step or three-step proposition are displayed by the elements of M^3, and finally all the propositions by M^5.

It is time to examine more closely the matrices themselves.

Boolean Matrices. Let R be the collection of $n \times n$ matrices $x, y, z, \cdots$ whose diagonal elements are all 1, whose other elements are either 0 or 1, subject to the usual laws of Boolean algebra.

THEOREM 1. *R is a lattice with 2^{n^2-n} elements.*

The fact that R is a lattice is not used in the following discussion, since we are interested in the product of these matrices. In terms of the lattice, the partial ordering is inclusion, and the meet and join are intersection and logical sum, respectively. Since our matrices have diagonal elements all equal to 1, there are a total of 2^{n^2-n} of them.

THEOREM 2. *Using addition and multiplication as the operations on the elements of R, we find that addition is closed, commutative, associative and possesses an identity (the usual multiplicative identity matrix serves as an additive identity). Multiplication is closed, associative, possesses an identity (the same one), but is not commutative. Multiplication is distributive with respect to addition.*

THEOREM 3. *If we define $x \subset y$ to mean that this relation holds for each pair of corresponding elements ($0 \subset 0$, $0 \subset 1$, $1 \subset 1$), then for any x, y, it follows that $x \subset xy$ and $x \subset yx$.*

Proof: We need only show that whenever an element x_{ij} of x is 1, then the corresponding elements of xy and yx is 1.

$$(xy)_{ij} = \sum_k x_{ik}y_{kj} = x_{i1}y_{1j} + \cdots + x_{ij}x_{jj} + \cdots + x_{in}y_{nj}.$$

But this sum is 1, since $y_{jj}=1$. Similarly $x \subset yx$.

THEOREM 4. *For any element x in R, $x^{n-1}=x^n=x^{n+1}= \cdots x^{n+p}$.* (The proof of this follows easily if given the logical interpretation of the previous section.)

We define $x \sim y$ to mean $x^{n-1}=y^{n-1}$. This is a reflexive, symmetric and transitive relation which divides R into equivalence classes. In the sense of the previous sections, the postulates described by x and y lead to the same propositions.

THEOREM 5. *If $x \sim y$, then $x+y \sim x$. (In the language analogous to that used in ordinary homogeneous linear differential equations, if x and y are two solutions of the equation $x^n=c$, then $x+y$ is also a solution.)*

(1) $(x+y)^{n-1}=x^{n-1}+y^{n-1}+x^{n-2}y+x^{n-3}yx+ \cdots$ where each of the missing terms is a product of $n-1$ factors arranged in some order. We will show that each of these missing terms is included in x^{n-1}, and, since $x^{n-1}=y^{n-1}$, the right hand side reduces to x^{n-1}. Consider a typical term, say $x^2y^3x^{n-1}y^{n-2} \cdot x^2y^3x^{n-4}y^{n-2} \subset x^2y^3x^{n-4}y^{n-1}$ (Theorem 3) $=x^2y^3x^{n-4}x^{n-1}=x^2y^3x^{n-1}=x^2y^3y^{n-1}=x^2y^{n-1}=x^2x^{n-1} =x^{n-1}$. Consequently, all the terms on the right hand side of (1) are included in x^{n-1}, and

$$(x+y)^{n-1} = x^{n-1}.$$

THEOREM 6. *If $x \sim y$, then $xy \sim x$ and $yx \sim x$.*

Proof: $x \subset xy$ and

$$x^{n-1} \subset (xy)^{n-1} = (xy)^{n-2}xy \subset (xy)^{n-2}xy^{n-1} = (xy)^{n-2}xx^{n-1} = (xy)^{n-2}x^{n-1}$$
$$= (xy)^{n-3}xyy^{n-1} = (xy)^{n-3}x^{n-1} = \cdots = x^{n-1}.$$

Since $x^{n-1} \subset (xy)^{n-1} \subset x^{n-1}$, then $(xy)^{n-1}=x^{n-1}$ and $xy \sim x$.

THEOREM 7. *For any x in R and y such that $x \subset y \subset x^{n-1}$, then $x \sim y$.*

Proof: Since $x \subset y$, $x^{n-1} \subset y^{n-1}$. But since $y \subset x^{n-1}$, $y^{n-1} \subset x^{2n-2}=x^{n-1}$. Therefore $x^{n-1} \subset y^{n-1} \subset x^{n-1}$, and $x^{n-1}=y^{n-1}$.

These theorems give us methods of finding matrices in a given equivalence class. The same kind of results are valid for Boolean matrices whose elements are Boolean functions and are useful for the electrical engineer who wishes to find the best way of devising a system of switches to perform a certain function. At least these theorems give him some of the possibilities, which he can examine and compare for simplicity, economy or reliability.

The economy motive is strong in mathematics. Given a set of propositions of the sort we are discussing, it would be desirable to find the smallest number of postulates whose assumptions guarantee these propositions. The last three theorems can yield other sets of postulates but in each case the newer sets include the old. It would be most desirable and a step in the right direction if the intersection of two equivalent matrices is in the same equivalence class. Unfortunately, this is not the case. A counter example is demonstrated by the two equivalent matrices

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

but the intersection

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

is not equivalent to these.

THEOREM 8. *If a set of postulates yields q theorems, there are at least 2^q different sets of equivalent postulates.*

According to the interpretation of Theorem 7, there will be 2^q matrices which include the matrix x and are included in the matrix x^{n-1}.

There is, however, one simple way of reducing the postulates of the system. If, for example, the postulates include $A_i \rightarrow A_j$, $A_j \rightarrow A_k$, $A_i \rightarrow A_k$, then the last of these is superfluous. A partial picture of the matrix will be

	Column j	Column k
Row i	1	1
Row j		1

and the 1 in position (ik) can be replaced by 0.

This would be quite tedious to check for each element in the matrix, but we notice that if the situation described above does exist, the vector product of row i with column k (with the usual arithmetic, not Boolean) will be at least 3. This is because the products $a_{ii}a_{ik}$, $a_{ij}a_{jk}$, and $a_{ik}a_{kk}$ are each 1. Conversely, if the vector product is at least 3, then the element a_{ik} is superfluous (in other words, the proposition a_{ik} is implied by the statements a_{ij} and a_{jk}) and the 1 in the position (ik) may be replaced by 0.

The procedure then is this: When a 1 occurs in position $(ik)(i\neq k)$, test the sum $\sum_j a_{ij}a_{jk}$. If it exceeds 2, replace the 1 by 0, then use the adjusted matrix in examining other possibilities.

For example, the matrix of Figure 2 is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Reduction yields the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

This is a system with five postulates and 10 propositions so that there are at least 2^{10} equivalent matrices. This last matrix admits no further reduction by this method since no row has more than two non-zero elements.

Yet if we examine the equivalent matrix associated with Figure 1, this reduction procedure does not change the matrix at all. Consequently, equivalent matrices do not reduce to the same matrix. Even more, the matrix to which a given matrix reduces may depend upon the order of examining the elements.

Conclusion. These Boolean matrices are applied mathematics in the sense that they can be applied to the study of mathematical systems. Some mathematicians may feel that such a study as this reduces mathematics to a system of formulas or techniques and robs it of some of its creativeness. Our contention is to the contrary—by reducing certain tasks in mathematics to a routine, more time and energy is available for the making of mathematics. Since it is possible to enlarge a simple system such as we have displayed by introducing new hypotheses such as $A_1\cap A_2$ or $A_1\cup(A_2\cap A_3)$ etc., there is still room for the imagination, the suitable definition, and the conjecture gained with the help of intuition and analogy. Indeed some of these conjectures may be quickly tested and proved or disproved. Witness the array of theorems which result from the group postulates. It is hardly likely that one could begin with a matrix and derive all of group theory. But it may be possible to test conjectures, derive theorems and test equivalence.

The matrices themselves form an interesting mathematical system with two operations, both closed and associative, the same identity serves both operations, one operation is commutative, one is not, and one distributive law is valid. To our knowledge such systems have not been studied. One might look for subsystems, homomorphism theorems, the analogue of ideals, units, primes, chains, polynomials, etc.

The definition of equivalence divides the matrices into distinct classes. These do not all have the same number of elements, since the identity matrix is in a class by itself, and we have displayed one class with at least 2^{10} elements. How many classes are there and how many matrices are in each class? Two of our theorems show that each class forms a closed system under two operations. If we add to any of these classes of equivalent matrices the identity matrix, we find an algebraic structure similar to R itself. How analogous is this to the concept of a subgroup or subring?

All the discussion to this point has been based on one principle of logic, namely that $A \rightarrow B$ and $B \rightarrow C$ implies that $A \rightarrow C$, or in slightly different language, that $A \cap B' = 0$ and $B \cap C' = 0$ implies that $A \cap C' = 0$. Another principle, not investigated here is that $A \cap B' = 0$ and $C \cap B' = 0$ together with $B \neq 0$ and $B \neq 1$ implies that $A'C' \neq 0$. It can be seen that 1 in position ij of the matrix xx^T (x^T is the transpose of x) means that $A_i' \cap A_j' \neq 0$. In practice, then, we can find some of the existence theorems.

Another principle of formal logic, namely that $BA' = 0$ and $BC \neq 0$ implies that $AC \neq 0$ is capable of matrix treatment and is being investigated.

THE FUNCTIONAL OPERATOR $Tf(x) = f(x+a)f(x+b) - f(x)f(x+a+b)$

H. W. GOULD, West Virginia University

The functional operator T defined in the title of this paper has some interesting properties when applied to well-known elementary functions in analysis. This comes about because of the fact that "generally speaking" the value obtained by the indicated calculations is intimately related to the product $f(a)f(b)$ and for apparently wide classes of functions this is the dominant term, the value of x having often only a minor effect on the result. For this reason we may think of the operator T as being a kind of cancellation operator, or an operator having a kind of invariance as to the value of x. In many cases where the value of $T_x f(x)$ does depend on x it is possible to normalize the operator T by either redefining f or by slightly changing the definition of T so that complete invariance is retained.

The writer's interest in the subject came about from the solution to a problem concerning Fibonacci numbers [7] shown him by John H. Biggs, a graduate student at West Virginia University. If we define the ordinary Fibonacci numbers by the recurrence relation $f_{n+2} = f_{n+1} + f_n$ and the initial conditions $f_1 = 1 = f_2$, is valid. To our knowledge such systems have not been studied. One might look for subsystems, homomorphism theorems, the analogue of ideals, units, primes, chains, polynomials, etc.

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$$T_n(f_n) = f_{n+a}f_{n+b} - f_nf_{n+a+b} = (-1)^n f_a f_b. \tag{1}$$

This formula may, of course, be quickly proved by induction. Notice that, in this instance at least, the absolute value of the result does not depend on n.

To further exhibit the nature of the operator T we state the following results, each of which can be checked in a routine fashion:

$$T(x) = ab, \tag{2}$$

$$T(c + dx) = d^2ab, \tag{3}$$

$$T(x^2) = a^2b^2 + 2abx(x + a + b), \tag{4}$$

$$T(\sin x) = T(\cos x) = \sin a \sin b, \tag{5}$$

$$-T(\sinh x) = T(\cosh x) = - \sinh a \sinh b, \tag{6}$$

$$T(K^{c+dx}) = 0 \qquad (c, d, K \text{ being constants}). \tag{7}$$

Here we have omitted the subscript x for simplicity, however we shall later introduce a more complicated symbolism to avoid confusion.

Verification of (5) or (6) makes interesting homework for a trigonometry class, and it may be of interest to examine a typical sequence of steps. Thus

$$\begin{aligned} T(\sin x) &= \sin (x + a) \sin (x + b) - \sin x \sin (x + a + b) \\ &= \sin (x + a)[\sin x \cos b + \cos x \sin b] \\ &\quad - \sin x[\sin (x + a) \cos b + \cos (x + a) \sin b] \\ &= \cos x \sin b \sin (x + a) - \sin x \sin b \cos (x + a) \\ &= \cos x \sin b[\sin x \cos a + \cos x \sin a] \\ &\quad - \sin x \sin b[\cos x \cos a - \sin x \sin a] \\ &= \sin a \sin b(\cos^2 x + \sin^2 x) = \sin a \sin b, \end{aligned}$$

the terms in x dropping out conveniently because of the most elementary trigonometric identity. The relations (5) should be compared with [5].

This then is the motivation which inspires one to see what else can be done with such an operator. In the present paper we present an overall view of this, with a few new results suggesting other avenues of study. Some of the questions we raise here will be dealt with elsewhere.

If we apply T to general powers of x we find a rather complicated result which does not seem to reduce to any really elegant form:

$$T(x^n) = \sum_{k=0}^{n} x^k \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-j} a^{n-j}b^{n-k+j} - \sum_{k=0}^{n} \binom{n}{k} x^{k+n}(a + b)^{n-k}. \tag{8}$$

However, the coefficient of x^n in $T(x^n)$, namely

$$\sum_{j=0}^{n} \binom{n}{j}^2 a^{n-j}b^j - (a + b)^n$$

has some connection with the Legendre polynomials since it is well known that these are expressible in the form

$$P_n(x) = \left(\frac{x-1}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{x+1}{x-1}\right)^k.$$

So it may be that known relations involving such polynomials can be used to express $T(x^n)$ into a more interesting form.

Much more can be learned by applying T to the classical polynomials of Legendre, Hermite, Jacobi, etc., and this has been carried out in a recent paper of Danese [3], where he considers the expression

$$F_{n+r}(x)F_{n+s}(x) - F_n(x)F_{n+r+s}(x)$$

in the case that $F_n(x)$ is an ultra-spherical polynomial $P_n^m(x)$, or a Jacobi polynomial, etc. Thus, normalizing $P_n^m(x)$ by writing $p_n^m = P_n^m(x)/P_n^m(1)$ Danese shows that p_n^m satisfies the interesting inequality

$$\text{(9)} \qquad \overset{m}{p}_{n+1}(x)\overset{m}{p}_{n+2}(x) - \overset{m}{p}_n(x)\overset{m}{p}_{n+3}(x) \geqq 0$$

provided that $m>0$, $0 \leqq x \leqq 1$, with equality holding only for $x=0$ or $x=1$.

In another paper Danese [2] has shown that the Hermite polynomials satisfy rather similar inequalities. Defining $H_n(x)$ by

$$H_n(x) = (-1)^n e^{x^2/2} D_x^n(e^{-x^2/2}).$$

Danese proves relations such as

$$[H_n'(x)]^2 - H_{n+1}''(x)H_{n-1}(x) \leqq 0, \qquad \text{all } x,\ n \geqq 1,$$

where the prime indicates differentiation with respect to x. This, as he shows, is related to the well-known inequality

$$H_n^2(x) - H_{n+1}H_{n-1}(x) > 0, \qquad \text{for all } x,\ n \geqq 1.$$

The left-hand side of this is also known to be explicitly equal to [8]

$$(n-1)! \sum_{k=0}^{n-1} \frac{H_k^2(x)}{k!}.$$

Study of the literature shows that expressions of the type given by the operator T when $a=1$ and $b=-1$ have most often been referred to as Turán expressions. The value of a study of such inequalities is shown by the fact that they arise in what is called Newton's test.

Newton's test may be stated as follows. Let C_n be the coefficient of z^n in the polynomial $f(z)$. Then $f(z)$ has imaginary zeroes if $C_{n-1}C_{n+1} - C_n^2 \geqq 0$. The interconnections with the results of Danese and the Fibonacci problem are therefore apparent.

Incidental to these remarks we must observe that the Hermite polynomials mentioned above differ from a more commonly defined polynomial of Hermite in recent literature. It is common to define

$$H_n(x) = (-1)^n e^{x^2} D_x^n e^{-x^2}. \tag{10}$$

We shall not take up inequalities here and shall restrict our attention more to explicit evaluations when the operator T is applied. To be sure, information about the positive or negative character of $T_n(H_n(x))$ would seem to depend on finding a simple explicit transformation of this which somehow removes the minus sign between the products. In the general case, for arbitrary a and b, this seems to require some complicated results in binomial coefficient summations.

It is known that the Hermite polynomials (10) satisfy the product formula

$$H_a(x)H_b(x) = \sum_{k=0}^{a+b+n} 2^k k! \binom{a}{k}\binom{b}{k} H_{a+b-2k}(x). \tag{11}$$

If we apply this to simplify the terms which occur in $T_n(H_n)$ we find that

$$T_n(H_n(x)) = \sum_{k=0}^{\min(a,b)} 2^k k! H_{2n+a+b-2k}(x) T_n\left\{\binom{n}{k}\right\}, \tag{12}$$

so that we are ultimately led to study T applied to the binomial coefficients. The problem which arises then is to simplify

$$T_x\left\{\binom{x}{n}\right\} = \binom{x+a}{n}\binom{x+b}{n} - \binom{x}{n}\binom{x+a+b}{n} \tag{13}$$

or discover inequalities for this expression. We shall leave this for discussion at length in another place. However the solution depends upon formulas similar to Nanjundiah's [6] formula

$$\binom{x}{m}\binom{y}{n} = \sum_{k=0}^{n} \binom{m-x+y}{k}\binom{n+x-y}{n-k}\binom{x+k}{m+n}.$$

Most of the difficulties which arise in a study of the general operator T come about because of the nonlinear nature of this operator. In fact explicit calculation shows that for two arbitrary functions f and g we have

$$\begin{aligned} T_x[f(x)+g(x)] &= T_x[f(x)] + T_x[g(x)] \\ &\quad + f(x+a)g(x+b) - f(x)g(x+a+b) \\ &\quad + g(x+a)f(x+b) - g(x)f(x+a+b). \end{aligned} \tag{14}$$

Only in special instances would the remainder terms vanish. Careful study shows that these terms are related in a special way to the functional equations satisfied by the functions f and g.

Suppose $g(x)=C$ is identically constant. Then (14) becomes

$$T_x[f(x)+C] = T_x[f(x)] + C\cdot S_x[f(x)], \tag{15}$$

where we introduce another operator S defined by

$$S_x[f(x)] = f(x+a) + f(x+b) - f(x) - f(x+a+b). \tag{16}$$

The relation (15) may be stated in another manner which would seem to give more insight into the relation between T and S:

$$S_x f(x) = \frac{T_x[f(x)+h] - T_x[f(x)]}{h}, \qquad h \neq 0. \tag{17}$$

This suggests that we call S a derived functional operator of T.

Although again we shall not develop any material on inequalities, it is not surprising that S comes up in that regard. Thus a formula of Jacobsthal, discussed by Carlitz in a recent paper [1] may be stated in the following form. Let $[x]$ denote the integral part of x. Then the result of Jacobsthal says that

$$\sum_{k=0}^{r-1} S_k \left\{ \left[\frac{k}{m} \right] \right\} \leqq 0, \tag{18}$$

provided that a, b are arbitrary integers and $m \geqq 1$, $r \geqq 1$. The interesting question, which the author has not been able to solve, is to find a similar result involving a summation with T instead of S.

We have tried to show so far something of the cancellation or invariant nature of the operator T and how it arises in connection with inequalities and other problems in classical polynomial theory. We desire now to attempt to generalize the operator T and make specific application of the results to obtain formulas involving the Fibonacci numbers. This has been motivated by an attempt to generalize (1) to a formula involving products of three f's instead of two.

Let us use the symbolism

$$\overset{2}{T}_{a,b}\{f(x)\} = T_x f(x) = f(x+a)f(x+b) - f(x)f(x+a+b), \tag{19}$$

for an operator applied to two parameters a, b and having a product of two functional values involved. Since we may think of this in terms of determinants in the form

$$T_x f(x) = \begin{vmatrix} f(x+a) & f(x) \\ f(x+a+b) & f(x+b) \end{vmatrix}$$

the thought occurs to extend T to three parameters by using three-dimensional determinants. If we imagine a cube with vertices labelled it appears reasonable to take something of the following form for a definition:

$$\begin{aligned} \overset{2}{T}_{abc} f(x) &= f(x+a)f(x+b+c) - f(x+b)f(x+a+c) \\ &\quad + f(x+c)f(x+a+b) - f(x)f(x+a+b+c). \end{aligned} \tag{20}$$

Defined this way we see that this operator is symmetrical in a and c so that

$$\overset{2}{T}_{abc} f(x) = \overset{2}{T}_{cba} f(x). \tag{21}$$

It is also readily verified by carrying out the calculations that the new operator is related to the original one by the formula

$$T^2_{abc}f(x) = T^2_{a-b,c}f(x+b) + T^2_{c,a+b}f(x). \tag{22}$$

If we apply this formula to (1) we obtain the relation

$$\begin{aligned} T^2_{abc}(f_n) &= f_{n+a}f_{n+b+c} - f_{n+b}f_{n+a+c} + f_{n+c}f_{n+a+b} - f_nf_{n+a+b+c} \\ &= (-1)^n f_c\{(-1)^b f_{a-b} + f_{a+b}\}. \end{aligned} \tag{23}$$

Next recalling the symmetry property (21), we find that

$$f_c\{(-1)^b f_{a-b} + f_{a+b}\} = f_a\{(-1)^b f_{c-b} + f_{c+b}\}. \tag{24}$$

This relation might be difficult to notice without some way of noticing the symmetry. Of course this might be done without having introduced the three-parameter operator. It is interesting to note that a formula rather like (24) is quoted in Dickson's history [4] as due to a certain Tagiuri who worked with a more general sequence than the Fibonacci sequence we have begun with, but presumably the symmetry principle would simplify some of the details. And of course the operator could be applied to something other than the Fibonacci numbers.

It may be of interest to observe that

$$T^2_{abc} \sin x = 2 \sin a \cos b \sin c, \tag{25}$$

which exhibits again the invariance principle, and the appearance of a cosine term in b instead of sine is intimately connected with the fact that in our definition (20) a minus sign appears on the term involving $f(x+b)$.

Let us now generalize to products of three functional values. The definition we shall take here is by no means the only possible one:

$$\begin{aligned} T^3_{abc}f(x) = {} & f(x+a)f(x+b)f(x+c) - f(x)f(x+a)f(x+b+c) \\ & + f(x)f(x+b)f(x+a+c) - f(x)f(x+c)f(x+a+b). \end{aligned} \tag{26}$$

By routine calculations it is readily verified that this operator may be expressed in terms of the original operator by the formula

$$T^3_{abc}f(x) = f(x+a)T^2_{b,c}f(x) - f(x+b)T^2_{c,a}f(x) + f(x+c)T^2_{a,b}f(x). \tag{27}$$

Therefore, without using any special properties of the Fibonacci numbers at all, we have by application of (27) and (1) the formula

$$\begin{aligned} & f_{n+a}f_{n+b}f_{n+c} - f_nf_{n+a}f_{n+b+c} + f_nf_{n+b}f_{n+a+c} - f_nf_{n+c}f_{n+a+b} \\ & \quad = (-1)^n(f_bf_cf_{n+a} - f_af_cf_{n+b} + f_af_bf_{n+c}). \end{aligned} \tag{28}$$

It might not be apparent from the left-hand member, but the right-hand member shows clearly that a simple recurrence relation holds for this expression. Indeed call the expression on the right inside the parentheses K_n. Then because each term involving f_{n+m} is multiplied by a constant, K_n satisfies the same recurrence relation as does f_n, namely $K_{n+2}=K_{n+1}+K_n$. Thus by knowing initial values of K_n and using elementary difference equation theory a formula could be written down for K_n, and the result, which is a little complicated, reads as follows:

$$(29)\qquad K_n = \frac{[2B-(1-\sqrt{5})A](1+\sqrt{5})^n + [A(1+\sqrt{5})-2B](1-\sqrt{5})^n}{2^{n+1}\sqrt{5}},$$

where

$$A = f_af_bf_c \quad\text{and}\quad B = f_bf_cf_{a+1} - f_cf_af_{b+1} + f_af_bf_{c+1}.$$

This is then one way to obtain a generalization of the original Fibonacci problem (1).

Now, one might imagine that an operator on four parameters and using the products of four functional values could be defined by extending (27), that is by letting

$$(30)\qquad \begin{aligned} \overset{4}{T}_{abcd}f(x) &= f(x+a)\overset{3}{T}_{bcd}f(x) - f(x+b)\overset{3}{T}_{acd}f(x) \\ &\quad + f(x+c)\overset{3}{T}_{abd}f(x) - f(x+d)\overset{3}{T}_{abc}f(x); \end{aligned}$$

however it is easily determined that this is identically zero. Thus one must make some change in the disposition of plus and minus signs in order to salvage anything from (30). There does not seem to be a unique way to go about this, however.

We have seen that the operator T can suggest new Fibonacci number relations, but all that we have said so far was restricted to operations in the real number system. It is possible to extend the operator T to more abstract settings, but, of course, we sacrifice some of the nice details which only hold for real numbers, or even integers.

Consider a system of elements subject to two operations $+$ and $\cdot$ with the ordinary rules of arithmetic holding except that multiplication is noncommutative. For example, let a, b, x be matrices. Then we find easily that $T(x)=ab+(ax-xa)$. In such a system then, the result would be simple and interesting only if the commutator $ax-xa=0$, which puts a severe restriction on things.

Another example of similarity to (4) is afforded by consideration of the so-called q-numbers. These were used by F. H. Jackson to define generalizations such as q-differences and q-binomial coefficients, which reduce to the familiar notions when $q\to 1$. By a q-number we shall mean the expression $[x]=(q^x-1)/(q-1)$. Then, of course, $\lim_{q\to 1}[x]=x$. The q-numbers satisfy non-linear laws such as $[a+b]=q^b[a]+[b]$ or

$$q^a[b]+q^b[a] = 2[a+b]-[a]-[b].$$

It is readily verified that

$$(31)\qquad T_x\{[x]\} = [a][b]+[x]\{[a+b]-[a]-[b]\}.$$

It is possible to consider the operator T in many general settings using the notation

$$T_x\{f(x)\} = f(x \cup a) \cap f(x \cup b) - f(x) \cap f(x \cup a \cup b), \tag{32}$$

where the operations $\cup$, $\cap$, and $-$ can have various meanings. For example this may be applied to lattices, or we can let

$$a \cap b = (a, b) = \text{g.c.d. } (a, b)$$
$$a \cup b = [a, b] = \text{l.c.m. } (a, b)$$

and work with the natural number system, using ordinary subtraction. Here we shall only consider a small example from set theory, and we shall take $\cup$ and $\cap$ to be the ordinary notions of union and intersection and for the subtraction operation let us define it by relative complementation to mean $a-b=a\cap b'$ where b' means the (absolute) complement of b. With f as the identity function, it is relatively easy to show that we may conclude for sets that

$$T_x(x) \subset a \cap b. \tag{33}$$

There is incidentally a dual operator T' associated with (32) and obtained by interchanging $\cup$ with $\cap$. In the case of the original operator T in application to the real number system its dual is

$$T'f(x) = f(ax) + f(bx) - f(x) - f(abx). \tag{34}$$

It is an interesting exercise to try to reduce this in the case of the Fibonacci numbers, and so find a dual to the original problem. By a separate part to that problem [7] it is easily seen that f_n must be a factor of $T'f_n$, that is to say

$$f_n \mid f_{an} + f_{bn} - f_n - f_{abn}.$$

But it is surely not apparent whether or not any simple relation exists involving f_a+f_b.

Finally we wish to remark that to generalize (1) to a relation involving the product of r terms $f_af_b \cdots f_m$ the apparently neatest way to achieve this is to work with Fibonacci numbers of scale of relation r. That is we require a sequence with recurrence relation like the following special instance. Let

$$\overset{r}{f}_{n+r} = \sum_{i=0}^{r-1} \overset{r}{f}_{n+i} \tag{35}$$

and assume initial conditions, for example, such as

$$\overset{r}{f}_i = 1, \qquad i = 1, 2, \cdots, r. \tag{36}$$

Then a generalization can be obtained for (1). Thus, let $r=3$. It is easy to verify by induction that (omitting the index $r=3$ in symbolism)

$$f_nf_{n+2}f_{n+4} + 2f_{n+1}f_{n+2}f_{n+3} - (f_{n+2})^3 - f_n(f_{n+3})^2 - f_{n+4}(f_{n+1})^2 = -4. \tag{37}$$

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THE MAXIMUM OF $\sum_{i;j=1}^{n} a_ib_j$

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If $a_1>a_2>\cdots>a_n>0$ and $b_1>b_2>\cdots>b_n>0$ are two ordered sequences, it is well known that the sum $\sum_{i;j=1}^{n} a_ib_j$ is a maximum when $i=j$. The following proof of this fact by induction, however, may be of some interest. If $n=1$, the problem is trivial. If $n=2$ we have $a_1>a_2>0$ and $b_1>b_2>0$, and two possible sums $a_1b_2+a_2b_1$ and $a_1b_1+a_2b_2$. Let

$$S = (a_1b_1 + a_2b_2) - (a_1b_2 + a_2b_1) = (a_1 - a_2)(b_1 - b_2) > 0.$$

Hence $a_1b_1+a_2b_2>a_1b_2+a_2b_1$, and $\sum_{i;j=1}^{2} a_ib_j$ is maximum when $i=j$.

Now assume that $\sum_{i;j=1}^{k} a_ib_j$ is maximum for $i=j$, and consider the two ordered sequences $a_1>a_2>\cdots>a_k>a_{k+1}>0$ and $b_1>b_2>\cdots>b_k>b_{k+1}>0$ and the sum $\sum_{i;j=1}^{k+1} a_ib_j$. For any $i=1,\cdots,k$, $a_i>a_{k+1}$ and $b_i>b_{k+1}$. Hence for any $i=1,\cdots,k$ there exist $p_i>0$ and $q_i>0$ such that $a_i=a_{k+1}+p_i$ and $b_i=b_{k+1}+q_i$; also $p_{k+1}=q_{k+1}=0$.

Therefore every term of $\sum_{i;j=1}^{k+1} a_ib_j$ is of the form $(a_{k+1}+p_i)(b_{k+1}+q_j)$. This leads to

$$\begin{aligned}\sum_{i;j=1}^{k+1} a_ib_j &= \sum_{i;j=1}^{k+1} (a_{k+1}+p_i)(b_{k+1}+q_j)\\ &= \sum_{i;j=1}^{k+1} \left[(a_{k+1}b_{k+1}) + (b_{k+1}p_i) + (a_{k+1}q_j) + (p_iq_j)\right].\end{aligned}$$

By definition $p_{k+1}=q_{k+1}=0$; therefore

$$\sum_{i;j=1}^{k+1} a_ib_j = a_{k+1}b_{k+1} + \sum_{i=1}^{k+1} b_{k+1}p_i + \sum_{j=1}^{k+1} a_{k+1}q_j + \sum_{i;j=1}^{k+1} p_iq_j. \tag{1}$$

Now, since the values of

$$a_{k+1}b_{k+1},\quad \sum_{i=1}^{k+1} b_{k+1}p_i,\quad \text{and}\quad \sum_{j=1}^{k+1} a_{k+1}q_j$$

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If $a_1 > a_2 > \cdots > a_n > 0$ and $b_1 > b_2 > \cdots > b_n > 0$ are two ordered sequences, it is well known that the sum $\sum_{i;j=1}^{n} a_i b_j$ is a maximum when $i = j$. The following proof of this fact by induction, however, may be of some interest. If $n = 1$, the problem is trivial. If $n = 2$ we have $a_1 > a_2 > 0$ and $b_1 > b_2 > 0$, and two possible sums $a_1 b_2 + a_2 b_1$ and $a_1 b_1 + a_2 b_2$. Let

$$S = (a_1 b_1 + a_2 b_2) - (a_1 b_2 + a_2 b_1) = (a_1 - a_2)(b_1 - b_2) > 0.$$

Hence $a_1 b_1 + a_2 b_2 > a_1 b_2 + a_2 b_1$, and $\sum_{i;j=1}^{2} a_i b_j$ is maximum when $i = j$.

Now assume that $\sum_{i;j=1}^{k} a_i b_j$ is maximum for $i = j$, and consider the two ordered sequences $a_1 > a_2 > \cdots > a_k > a_{k+1} > 0$ and $b_1 > b_2 > \cdots > b_k > b_{k+1} > 0$ and the sum $\sum_{i;j=1}^{k+1} a_i b_j$. For any $i = 1, \cdots, k$, $a_i > a_{k+1}$ and $b_i > b_{k+1}$. Hence for any $i = 1, \cdots, k$ there exist $p_i > 0$ and $q_i > 0$ such that $a_i = a_{k+1} + p_i$ and $b_i = b_{k+1} + q_i$; also $p_{k+1} = q_{k+1} = 0$.

Therefore every term of $\sum_{i;j=1}^{k+1} a_i b_j$ is of the form $(a_{k+1} + p_i)(b_{k+1} + q_j)$. This leads to

$$\sum_{i;j=1}^{k+1} a_i b_j = \sum_{i;j=1}^{k+1} (a_{k+1} + p_i)(b_{k+1} + q_j)$$

$$= \sum_{i;j=1}^{k+1} \left[(a_{k+1} b_{k+1}) + (b_{k+1} p_i) + (a_{k+1} q_j) + (p_i q_j)\right].$$

By definition $p_{k+1} = q_{k+1} = 0$; therefore

$$\sum_{i;j=1}^{k+1} a_i b_j = a_{k+1} b_{k+1} + \sum_{i=1}^{k+1} b_{k+1} p_i + \sum_{j=1}^{k+1} a_{k+1} q_j + \sum_{i;j=1}^{k+1} p_i q_j. \tag{1}$$

Now, since the values of

$$a_{k+1} b_{k+1}, \quad \sum_{i=1}^{k+1} b_{k+1} p_i, \quad \text{and} \quad \sum_{j=1}^{k+1} a_{k+1} q_j$$

remain constant whether or not $i=j$, the only "variable" term on the right-hand side in (1) is $\sum_{i;j=1}^{k+1} p_i q_j$.

By definition $a_i = a_{k+1} + p_i$ and $b_i = b_{k+1} + q_i$ for $i=1, \cdots, k$ and this implies that, since $a_i > a_{i+1}$ and $b_i > b_{i+1}$ for $i=1, \cdots, k-1$, $a_{k+1}+p_i > a_{k+1}+p_{i+1}$ and $b_{k+1}+q_i > b_{k+1}+q_{i+1}$ or that $p_i > p_{i+1}$ and $q_i > q_{i+1}$ for $i=1, \cdots, k-1$. Hence the p's and q's can be ordered as $p_1 > p_2 > \cdots > p_k > 0$ and $q_1 > q_2 > \cdots > q_k > 0$, and $p_{k+1} = q_{k+1} = 0$.

Case 1. If one term of $\sum_{i;j=1}^{k+1} p_i q_j$ is $p_{k+1}q_{k+1} = 0$, then

$$\sum_{i;j=1}^{k+1} p_i q_j = \sum_{i;j=1}^{k} p_i q_j$$

which is maximum when $i=j$ by the induction hypothesis.

Case 2. If $p_{k+1}q_s = 0$ and $q_{k+1}p_t = 0$ (where $1 \leqq s, t \leqq k$) are terms of $\sum_{i;j=1}^{k+1} p_i q_j$, it is clear that

$$\sum_{i;j=1}^{k+1} p_i q_j = \sum_{i;j=1}^{k} p_i q_j - p_t q_s.$$

Now, by the induction hypothesis, $\sum_{i;j=1}^{k} p_i q_j$ is maximum when $i=j$, or in other words

$$\sum_{i=1}^{k} p_i q_i \geqq \sum_{i;j=1}^{k} p_i q_j.$$

Since $p_t q_s > 0$, it follows that

$$\sum_{i=1}^{k} p_i q_i > \sum_{i;j=1}^{k} p_i q_j - p_t q_s.$$

Using these, plus the fact that

$$\sum_{i=1}^{k+1} p_i q_i = \sum_{i=1}^{k} p_i q_i \text{ (since } p_{k+1} = q_{k+1} = 0),$$

it follows that

$$\sum_{i=1}^{k+1} p_i q_i = \sum_{i=1}^{k} p_i q_i > \sum_{i;j=1}^{k} p_i q_j - p_t q_s = \sum_{i;j=1}^{k+1} p_i q_j, \quad \text{or} \quad \sum_{i=1}^{k+1} p_i q_i > \sum_{i;j=1}^{k+1} p_i q_j,$$

or in other words $\sum_{i;j=1}^{k+1} p_i q_j$ is maximum when $i=j$. It has then been shown that

$$\sum_{i;j=1}^{k+1} a_i b_j = a_{k+1}b_{k+1} + \sum_{i=1}^{k+1} b_{k+1}p_i + \sum_{j=1}^{k+1} a_{k+1}q_j + \sum_{i;j=1}^{k+1} p_i q_j$$

is maximum when $i=j$, and the proof is complete.

FOUR EQUAL TRITANGENT CIRCLES

LEON BANKOFF, Los Angeles, California

The radii of the circles (0), (0_1), (0_2) are R, R_1, R_2, respectively, where $R=2R_1=2R_2$.

In the upper semicircle, the radius of the circle (W_n) in the chain is r_n. Now, $r_n=RR_1R_2/(RR_1+n^2R_2^2)$, so the radius of the cross hatched circle is

$$r_4 = (R^3/4)/(R^2/2 + 16R^2/4) \quad \text{or} \quad R/18.$$

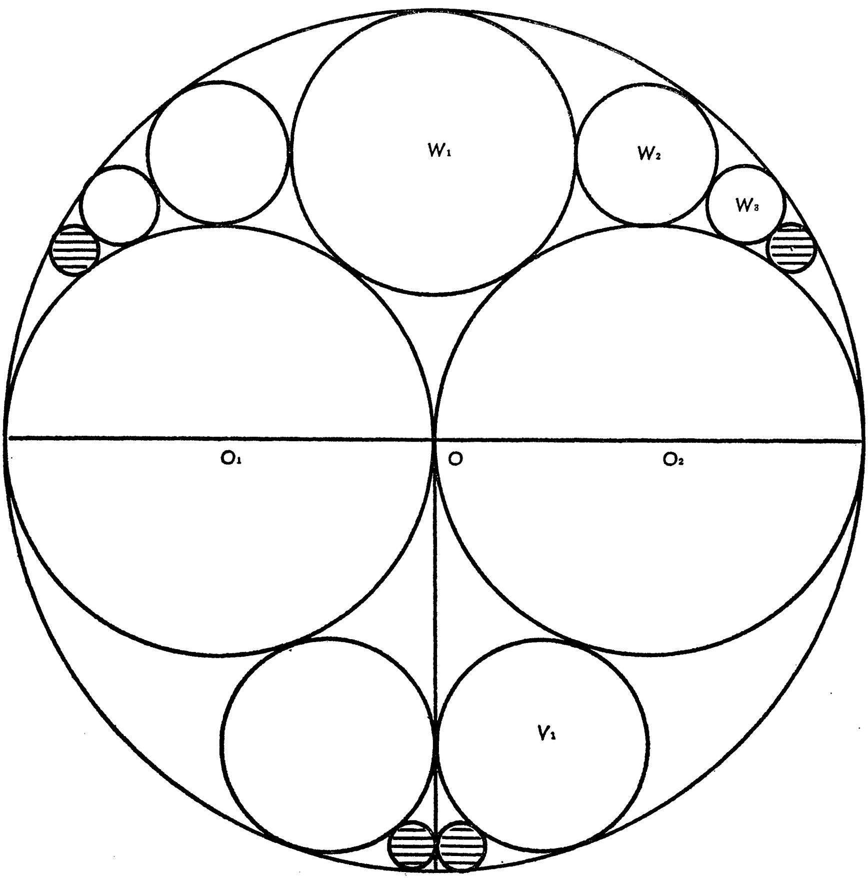
In the lower semicircle, the radius, ρ_n, of the circle (V_n) in the chain is given by the remarkable formula

$$\rho_n = 4R_1^n R_2/[(\sqrt{R} + \sqrt{R_2})^n + (\sqrt{R} - \sqrt{R_2})^n]^2.$$

For the cross hatched circle, $n=2$, so

$$\rho_2 = (R^3/2)/[(\sqrt{R} + \sqrt{R/2})^2 + (\sqrt{R} - \sqrt{R/2})^2]^2 \quad \text{or} \quad R/18.$$

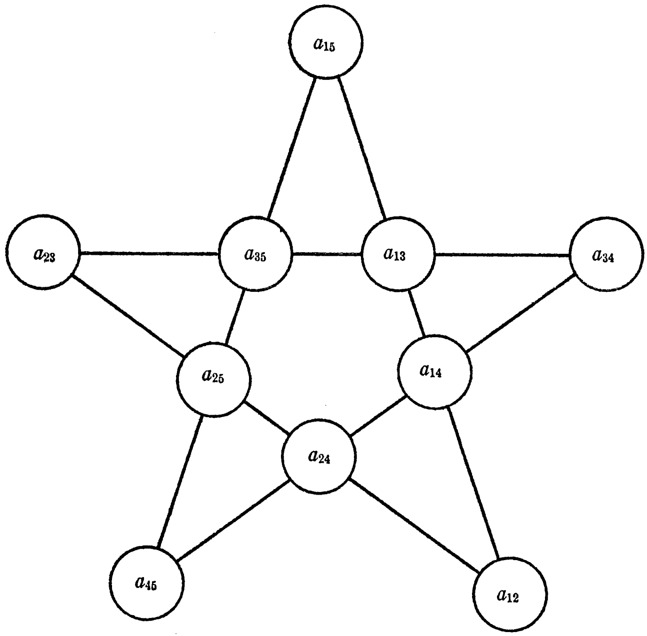
Therefore, $r_4=\rho_2$.



COMMENT ON "A MAGIC PENTAGRAM"

D. I. A. COHEN, Midwood High School, Brooklyn, N. Y.

In the September-October 1962 issue of this MAGAZINE, p. 228, C. W. Trigg gave a representation of 1962 in the form of a Magic Pentagram. The question then arises of which numbers can be written in this form. Let us number the vertices as shown in the figure.



We see that if a number K can be written in the above form then $K+4n$ can also be written in that form by adding the number n to each element a_{ij} for K.

Assume that the sum of the elements along any line is K. Subtract the number in position a_{13} from each element; now the sum is $K-4a_{13}=K'$.

Let $a_{15}=a$, $a_{14}=b$, $a_{24}=c$, $a_{25}=d$, and $a_{35}=e$.

Along line a_{15} to a_{12} we get $a_{12}=K'-a-b$;

Along line a_{12} to a_{23} we get $a_{23}=a+b-c-d$;

Along line a_{23} to a_{34} we get $a_{34}=K'-a-b+c+d-e$;

Along line a_{34} to a_{45} we get $a_{45}=a-2c-d+e$, but along line a_{15} to a_{45} we get $a_{45}=K'-d-e-a$.

Therefore $K'-d-e-a=a-2c-d+e$ and $K'=2a-2c+2e$.

So that we see K' is even and then so is K. Therefore all numbers so expressible are even.

By subtracting 490 from each of the numbers given in the article we get a magic pentagram for $1962-4\cdot 490=2$. This is: $a_{15}=12$, $a_{13}=-7$, $a_{14}=-9$, $a_{12}=6$, $a_{24}=-6$, $a_{25}=-8$, $a_{23}=10$, $a_{35}=-10$, $a_{34}=9$, and $a_{45}=8$. Therefore all numbers of the form $2+4n$ can be written as a magic pentagram.

A magic pentagram for 4 could be one with all the elements equal to one. But if we make the restriction that all the numbers are different we get our magic pentagram by doubling the value of each element in the above representation for two. Therefore all numbers of the form $4n$ are expressible in this form.

So a necessary and sufficient condition that a number be expressible as a magic pentagram is that it be even.

By adding 490 to each element of the magic pentagram for 4 we get one for 1964.

EXISTENCE THEOREMS

THOMAS P. HAGGERTY, Wheeling College, West Virginia

In order to impress upon students the importance of Gauss' Fundamental Theorem of Algebra, it is necessary to exhibit a fairly simple equation for which no solution exists. The following is especially helpful:

$$x - \sqrt[3]{(1 + x^3)} = 0.$$

"Solving" this we get:

$$\begin{aligned} x &= \sqrt[3]{(1 + x^3)}, \\ x^3 &= 1 + x^3, \\ 0 &= 1. \end{aligned}$$

The resolution of this "paradox" in ensuing discussion brings out very nicely the underlying assumption that a solution did, in fact, exist; which position is subsequently shown to be untenable.

BOUNDEDNESS OF THE SEQUENCE $(1+1/n)^n$

BEVAN K. YOUSE, Emory University

It is well known that the sequence $a_n = (1+1/n)^n$ is monotonic increasing and bounded above; hence,

$$\lim_{n\to\infty} a_n$$

exists. The purpose of this note is to give a simple proof that $a_n < 2.7916\underline{6} \cdots$.

Using the sum formula for a geometric series, we have

$$\sum_{K=4}^{n} \frac{1}{2^K} = \frac{1}{2^3} - \frac{1}{2^n}.$$

Hence,

$$\sum_{K=4}^{n} \frac{1}{2^K} < \frac{1}{8} \qquad \text{for all } n.$$

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Hence,

$$\sum_{K=4}^{n} \frac{1}{2^K} < \frac{1}{8} \qquad \text{for all } n.$$

Adding $(1/2!+1/3!)$ to both sides of the inequality,

$$\frac{1}{2!}+\frac{1}{3!}+\sum_{K=4}^{n}\frac{1}{2^K}<\frac{19}{24}.$$

Since $n!>2^n$ for $n\geqq 4$,

$$\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots+\frac{1}{n!}<\frac{19}{24}.$$

Thus,

$$\frac{n(n-1)}{n^2}\cdot\frac{1}{2!}+\frac{n(n-1)(n-2)}{n^3}\cdot\frac{1}{3!}+\frac{n(n-1)(n-2)(n-3)}{n^4}\cdot\frac{1}{4!}+\cdots$$
$$+\frac{n!}{n^n}\cdot\frac{1}{n!}<\frac{19}{24}\cdot$$

Adding $1+n(1/n)=2$ to both sides of inequality,

$$\sum_{K=0}^{n}{}_nC_K\,\frac{1}{n^K}<2\,\frac{19}{24}.$$

Since,

$$(1+1/n)^n=\sum_{K=0}^{n}{}_nC_K\,\frac{1}{n^K},\qquad (1+1/n)^n<2\frac{19}{24}=2.7916\underline{6}\cdots.$$

COMMENT ON THE PAPER "SOME PROBABILITY DISTRIBUTIONS AND THEIR ASSOCIATED STRUCTURES"

L. CARLITZ, Duke University

In the paper with the above title [2] in the May, 1963 issue of this MAGAZINE, N. R. Dilley has considered arrays of the type

$$\begin{array}{lllllllll}
1 \\
1 & 1 & 1 \\
1 & 2 & 3 & 2 & 1 \\
1 & 3 & 6 & 7 & 6 & 3 & 1 \\
1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1.
\end{array}$$

The mode of formation is obvious. If we let a_{nr} denote the element in the nth row and rth column, then we have the recurrence

$$a_{n+1,r}=a_{n,r-2}+a_{n,r-1}+a_{nr}. \tag{1}$$

This formula, together with the boundary conditions

$$a_{00}=1,\qquad a_{0r}=0\qquad (r\neq 0), \tag{2}$$

Adding $(1/2!+1/3!)$ to both sides of the inequality,

$$\frac{1}{2!}+\frac{1}{3!}+\sum_{K=4}^{n}\frac{1}{2^K}<\frac{19}{24}.$$

Since $n!>2^n$ for $n\geqq 4$,

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Thus,

$$\frac{n(n-1)}{n^2}\cdot\frac{1}{2!}+\frac{n(n-1)(n-2)}{n^3}\cdot\frac{1}{3!}+\frac{n(n-1)(n-2)(n-3)}{n^4}\cdot\frac{1}{4!}+\cdots$$
$$+\frac{n!}{n^n}\cdot\frac{1}{n!}<\frac{19}{24}\cdot$$

Adding $1+n(1/n)=2$ to both sides of inequality,

$$\sum_{K=0}^{n}{}_nC_K\frac{1}{n^K}<2\frac{19}{24}.$$

Since,

$$(1+1/n)^n=\sum_{K=0}^{n}{}_nC_K\frac{1}{n^K}, \qquad (1+1/n)^n<2\frac{19}{24}=2.7916\underline{6}\cdots.$$

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The mode of formation is obvious. If we let a_{nr} denote the element in the nth row and rth column, then we have the recurrence

$$a_{n+1,r}=a_{n,r-2}+a_{n,r-1}+a_{nr}. \tag{1}$$

This formula, together with the boundary conditions

$$a_{00}=1, \qquad a_{0r}=0 \qquad (r\neq 0), \tag{2}$$

uniquely determines the a_{nr}. Note in particular that

$$a_{nr} = 0 \qquad (r < 0, r > 2n).$$

In order to obtain an explicit formula for a_{nr} the following method [1] is efficacious. Let E^{-1} denote the operator such that

$$E^{-1}f(r) = f(r-1).$$

Then (1) becomes $a_{n+1,r} = (E^{-2} + E^{-1} + 1)a_{nr}$, which yields

$$a_{nr} = (E^{-2} + E^{-1} + 1)^n a_{0r}.$$

Expanding $(E^{-2}+E^{-1}+1)^n$ by the multinomial theorem and making use of (2), we get

$$\begin{aligned} a_{nr} &= \sum_{2j\le r} \frac{n!}{j!(r-2j)!(n-r+j)!} = \sum_{2j\le r} \binom{n}{r-j}\binom{r-j}{j} \\ &= \sum_{2j\le r} \binom{n}{j}\binom{n-j}{r-2j}. \end{aligned} \tag{3}$$

It does not seem possible to sum the series on the right of (3). We may however notice the following generating functions:

$$\begin{aligned} &\sum_{r=0}^{2n} a_{nr}y^r = (1+y+y^2)^n, \qquad \sum_{n=0}^{\infty} a_{nr}x^n = \sum_{2j\le r}\binom{r-j}{j}\frac{x^{r-j}}{(1-x)^{r-j+1}}, \\ &\sum_{n=0}^{\infty} a_{nr}\frac{x^n}{n!} = e^x \sum_{2j\le r}\frac{x^{r-j}}{j!(r-2j)!}, \qquad \sum_{n=0}^{\infty}\sum_{r=0}^{2n} a_{nr}x^n y^r = \frac{1}{1-x(1+y+y^2)}, \\ &\sum_{n=0}^{\infty}\sum_{r=0}^{2n} a_{nr}\frac{x^n}{n!}y^r = e^{x(1+y+y^2)}. \end{aligned} \tag{4}$$

This suggests relationships involving various special functions.

We remark that (4) implies the symmetry relation

$$a_{nr} = a_{n,2n-r} \quad (0 \leqq r \leqq 2n). \tag{5}$$

We remark also that

$$\sum_{n=0}^{\infty} a_{nn}x^n = (1-2x-3x^2)^{-1/2} \tag{6}$$

as follows without much difficulty from (3). This special case was studied in great detail by Euler [3].

Supported in part by NSF grant G16485.

References

1. L. Carlitz, On arrays of numbers, Amer. J. Math., **54** (1932) 739–752.

2. N. R. Dilley, Some probability distributions and their associated structures, this MAGAZINE, **36** (1963) 175–179.

3. L. Euler, Observationes analyticae, Opera omnia, series prima, Leipzig and Berlin, **15** (1927) 50–69.

E20. If

$$\sqrt{\left\{a+\frac{b}{c}\right\}} = a\sqrt{\frac{b}{c}},$$

then $ac+b=a^2b$ or $ac=b(a^2-1)$. Now a is prime to a^2-1, so divides b and a^2-1 divides c. This problem is a special case in which $a=3$, $b=3$, $c=8=a^2-1$. Other mixed numbers in this same curious category, $a+a/(a^2-1)$, are $2\frac{2}{3}$, $4\frac{4}{15}$, $5\frac{5}{24}$, $6\frac{6}{35}$, etc.

E21. One explanation is that when an indefinite integral occurs, the equality is to be regarded as an equivalence relation such that two functions are equivalent when they differ by a constant.

(Falsies on page 62)

Answers

A327. $a^2+b^2-2ab\lambda\cos C=(1-\lambda)a^2+(1-\lambda)b^2+\lambda c^2$. Consequently, $a=b=c$ unless $\lambda=1/2$ for which case the equations are identically satisfied.

A328. Let

$$\phi(a) = \int_0^\infty \frac{1-e^{-at}}{t^m}\,dt$$

then

$$\phi'(a) = \int_0^\infty \frac{e^{-at}}{t^{m-1}}\,dt = \frac{1}{a^{2-m}}\,\Gamma(2-m).$$

Hence

$$\phi(1) = \frac{1}{m-1}\,\Gamma(2-m) = -\,\Gamma(1-m).$$

This procedure can be extended to the integrals of the form

$$\int_0^\infty \left(1-t+\frac{t^2}{2!}-\cdots-e^{-t}\right)\frac{dt}{t^r}\,.$$

A329. The given number lies between the squares of

$$n^2+n \quad\text{and}\quad n^2+n+1.$$

A330. We have

$$\begin{aligned} 0 = (1-1)^r &= \sum_{s=0}^{r} (-1)^s \binom{r}{s} \\ &= \sum_{s=0}^{n} (-1)^s \binom{r}{s}, \qquad r = 1, 2, \cdots, n. \end{aligned}$$

Therefore $P(x)=(1-x)(2-x)\cdots(n-x)/n!$

(Quickies on page 62)

DISTANCE BETWEEN TWO POINTS ON A SPHERE

CURT MARCUS, System Development Corporation, Santa Monica, Calif.

Both of the two usual methods for computing the shortest great circle distance d between two points, expressed by latitude ϕ and longitude λ, on a unit sphere have disadvantages for computer usage. For instance, the Law of Cosines for spherical triangles yields inaccurate results for small distances, where more precision is often desired, since it requires computing the inverse cosine of a number close to one. Napier's Analogies are time consuming by requiring the calculation of intermediate angles.

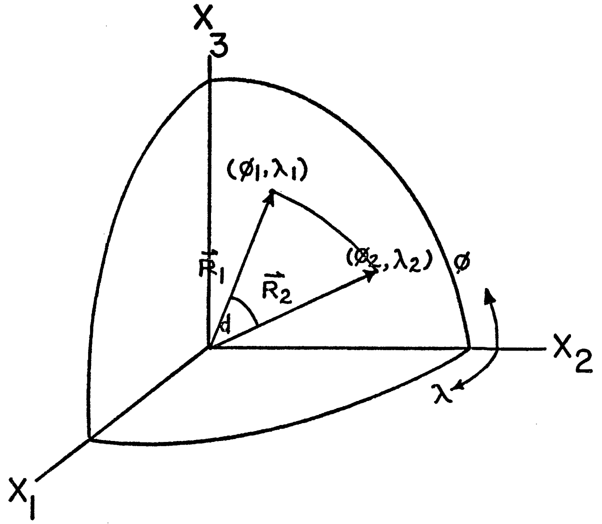


FIG. 1

The following equation, easily derived with the aid of Figure **1**, avoids the above mentioned difficulties:

$$\sin\frac{d}{2} = \frac{\sqrt{(\vec{R_2} - \vec{R_1})\cdot(\vec{R_2} - \vec{R_1})}}{2} \qquad 0 \leqq d \leqq \pi$$

where

$$\vec{R_1} = (\cos\phi_1 \sin\lambda_1, \cos\phi_1 \cos\lambda_1, \sin\phi_1)$$

and

$$\vec{R_2} = (\cos\phi_2 \sin\lambda_2, \cos\phi_2 \cos\lambda_2, \sin\phi_2).$$

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles 29, California.

PROPOSALS

537. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Determine the relative positions of an equilateral triangle and a square inscribed in the same circle so that their common area shall be an extremum.

538. *Proposed by Andrzej Makowski, Warsaw, Poland.*

Prove that for every integer $n>1$, the inequality $[d(n)]^2\phi(n)>\sigma(n)$ holds where $d(n)$ denotes the number of positive divisors of (n), $\sigma(n)$ denotes their sum, and $\phi(n)$ is Euler's totient function.

539. *Proposed by L. Carlitz, Duke University.*

Let P be a point inside the triangle ABC whose distances from the sides are x, y, and z. Let K denote the area and R the circumradius of ABC. Show that $xyz \leqq (2/27)(K^2/R)$ with equality holding only when P is the centroid of ABC.

540. *Proposed by Leo Moser, University of Alberta.*

Show that if the $n!$ terms in the expansion of an nth order determinant with positive elements a_{ij} have the same absolute value, then there exists a set of numbers $b_1, b_2, \cdots, b_n$ such that $a_{ij}=b_i \cdot b_j$, $i, j=1, 2, \cdots, n$.

541. *Proposed by J. Barry Love, Eastern Baptist College, Pennsylvania.*

Let p be a prime, and let n be the smallest positive quadratic residue (mod p). Show that $n<1/2+\sqrt{p}$.

542. *Proposed by Brother U. Alfred, St. Mary's College, California.*

Given a set of circles of radius R on an extended surface with their centers at the corners of a network of squares of side d. Let a ring of radius r be tossed on the surface. If $r \leqq R$ and $2(R+r) \leqq d$, what is the probability that the ring will touch one of the circles?

543. *Proposed by Murray S. Klamkin, State University of New York at Buffalo.*

If $\int_a^b [F(x)-x^r]^2 dx=\lambda^2$, find upper and lower bounds for $\int_a^b [F(x)]^2 dx$.

(Note: For a class of similar problems, see J. L. Synge, *The Hypercircle in Mathematical Physics*, p. 82.)

SOLUTIONS

Murder!

516. [May 1963]. *Proposed by Maxey Brooke, Sweeny, Texas.*

Six men, Adams, Brown, Jones, McCall, Smith, and Williams are sitting at equal intervals around a circular table.

That is, five of them are sitting. The sixth is slumped in that so-dead position corpses assume. One of the men at the table is his murderer. We know the following facts:

1. Jones sits to the left of the uncle of the man just across the table from him.
2. The murdered man had no relatives.
3. Adams asks McCall, who is sitting next to him, for a cigarette.
4. The murderer does not sit next to the uncle or the nephew, but the murdered man sits between them.
5. The man on Smith's right, who is not Jones, sits next to the murderer.
6. Adams sits directly across the table from Williams who is smoking nervously.

Who was killed by whom?

Solution by Thomas R. Hamrick, San Quentin, California.

Starting with Jones and proceeding in order to his right, label the seats J, T, D, Q, X, and Y.

From 1., and 4., Y is the killer and Jones is alive, as are Adams, McCall, and Williams (3., and 6.). Smith is not in D (5.), thus he is alive. The victim, in D, is Brown. This eliminates Adams, Williams, Jones, and Smith as suspects (4., 5., and 6.). Therefore, McCall killed Brown. Condition 2. is extraneous.

Also solved by Marc Aronson, University of Florida; Merrill Barneby, University of North Dakota; Joseph B. Bohac, St. Louis, Missouri; J. L. Brown, Jr., Pennsylvania State University; Daniel I. A. Cohen, Princeton University; Monte Dernham, San Francisco, California; Harry M. Gehman, SUNY at Buffalo, New York; Eugene D. Gingerich, Santa Barbara City College, California; Daniel L. Hansen, Westmar College, Iowa; Gilbert Labelle, Université de Montréal, Canada; Gerald R. Rising, Board of Education, Norwalk, Connecticut; David L. Silverman, Beverly Hills, California; Ralph N. Vawter, St. Mary's College, California; Hazel S. Wilson, Jacksonville University, Florida; Brother Louis F. Zirkel, Archbishop Molloy High School, Jamaica, New York; and the proposer.

Parabolic Areas

517. [May 1963] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let F and d be the focus and directrix of a parabola. If M and N are any two points on the parabola and M', N' are their respective projections on d, show that

$$\frac{\text{Area } FMN}{\text{Area } N'M'MN} = \text{Constant.}$$

I. *Solution by Francis D. Parker, University of Alaska.*

Using a focal length of F and orienting the directrix on the x-axis and the focus on the y-axis, we may use $y=x^2/4F+F$ as the equation of the parabola. If the abscissas of M and N are a and b, respectively, straightforward calculations yield

$$\text{Area } MM'N'N = \int_a^b y\,dx = \frac{b-a}{12F}\left[12F^2 + a^2 + ab + b^2\right]$$

and

$$\text{Area } FMN = \frac{b-a}{24F}\left[12F^2 + a^2 + ab + b^2\right].$$

Hence, the ratio of the areas is independent of F, a, and b, and is equal to 1/2.

II. *Solution by Joel Kugelmass, Stanford University and the National Bureau of Standards.*

It is clear that any parabola $f_1(x)$ can be transformed into another parabola $f_2(x)$ by applying a projective transformation P, an orthogonal transformation O and suitable rotations and translations. All of these transformations preserve the ratio of the area of the triangular region to that of the trapezoidal region. Hence we may transform any parabola to $y=x^2$. If we transform again so that $\lim M-N=0$, the areas clearly approach the length of their altitudes which in turn approaches p, the distance from the focus to the center. Now the function $z=(p+\epsilon_1)/(p+\epsilon_2)$, where the divisor and dividend are the areas of the regions, is monotone after a sufficient number of transformations ($\epsilon_1+\epsilon_2<\delta$) and hence approaches the limit. Now as all of the ratios are the same under the given transformations, the original ratio equals a constant and the theorem is proved.

Also solved by Josef Andersson, Vaxholm, Sweden; Michael J. Pascual, Watervliet Arsenal, New York; Hazel S. Wilson, Jacksonville University, Florida; and the proposer.

Dermott A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; P. R. Nolan, Department of Education, Dublin, Ireland; and Brother Louis F. Zirkel, Archbishop Molloy High School, Jamaica, New York; each pointed out that the proposal is incorrect if the figures FMN and N'M'MN are considered to be the rectilinear areas instead of areas bounded by the arc of the parabola, MN.

One incorrect solution was received.

Unique Determiner

518. [May 1963] *Proposed by Murray S. Klamkin, State University of New York at Buffalo.*

Show that an integer is determined uniquely from a knowledge of the product of all its divisors.

I. *Solution and comments by Leo Moser, University of Alberta.*

By pairing a divisor d of n with its complementary divisor n/d (and leaving $\sqrt{n}$, if it is a divisor, unpaired) we see that the geometric mean of the divisors of n is $\sqrt{n}$ and hence, if $\tau(n)$ denotes the number of divisors of n,

$$\prod_{d|n} d = n^{\tau(n)/2}. \tag{1}$$

We therefore need to show that

$$n^{\tau(n)} = m^{\tau(m)} \text{ implies that } n = m. \tag{2}$$

We will show more generally that if $f(n)$ is an arithmetic function for which

$$m \mid n \text{ implies } f(m) \leqq f(n) \text{ then} \tag{3}$$

$$n^{f(n)} = m^{f(m)} \text{ implies } n = m. \tag{4}$$

Proof of (4): Clearly n and m have the same prime factors. Suppose that

$$n = P_1^{\alpha_1} \cdots P_k^{\alpha_k} \quad \text{and} \quad m = P_1^{\beta_1} \cdots P_k^{\beta_k} \tag{5}$$

are the prime power representations of n and m. Comparing the exponents of P_1 in n and m we find

$$\alpha_1 f(n) = \beta_1 f(m). \tag{6}$$

Similarly

$$\alpha_2 f(n) = \beta_2 f(m). \tag{7}$$

From (6) and (7) we find

$$\frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2} \tag{8}$$

and similarly we find that

$$\frac{\alpha_2}{\beta_1} = \frac{\alpha_2}{\beta_2} = \cdots = \frac{\alpha_k}{\beta_k}\,. \tag{9}$$

If this common ratio is 1 we are done. If not, assume without loss of generality that it exceeds 1. Then $m|n$ and by (3) $f(m) \leqq f(n)$. Also $m < n$ so that $n^{f(n)} > m^{f(m)}$ and the result is established.

We note that special cases of suitable $f(n)$ include

$$f(n) = \phi(n) = \sum_{\substack{(a,n)=1 \\ a \leqq 1}} 1, \qquad f(n) = \sigma(n) = \sum_{d|n} d,$$

and $f(n) = w(n)$, where $w(n)$ is the number of distinct prime divisors of n.

Somewhat related to the fact that $n^{\phi(n)} = m^{\phi(m)}$ implies $n = m$ is the fact that $n\phi(n) = m\phi(m)$ implies $n = m$. This appears as a problem in *An Introduction to the Theory of Numbers* by Niven and Zuckerman. On the other hand we note that the corresponding result is not true for ϕ replaced by σ. In fact $12\sigma(12) = 14\sigma(14)$ and more generally, if $(a, 42) = 1$ then $12a\sigma(12a)/14\sigma(12a) = 14\sigma(14a)$.

Let us call a solution of $n\sigma(n) = m\sigma(m)$, $n \neq m$, primitive, if it cannot be obtained from a smaller solution by multiplying through by some factor. We have not been able to decide whether $n\sigma(n)=m\sigma(m)$, $n\neq m$ has infinitely many primitive solutions.

II. *Solution by Joel Kugelmass, Stanford University and the National Bureau of Standards.*

Let $\alpha(A)$ be the product of the divisors of A and $P_1^{\alpha_1}\cdots P_r^{\alpha_r}$ the canonical decomposition of A. Associate with A the point $(\alpha_1,\cdots,\alpha_r)$ in the r dimensional rectangle containing points $P_i(x_1,\cdots,x_r)$ with $0\leqq x_i\leqq\alpha_i$ and $1\leqq i\leqq\tau(A)$. Also $\alpha(A)\rightarrow(\alpha_1',\cdots,\alpha_r')$ which is equal to the vector sum $s(A)=\sum_{i=1}^{\tau(A)}P_i$. Now consider the $\tau(A)$ dimensional space P' with the points P_i as the vector basis. If $T\colon P_i\rightarrow P_i'$, the image of $\alpha(A)$ in P maps into P' at a point $Q=s(A)$. Q must be unique since $\{P_i\}$ is a vector basis of P' and each of the vectors form a linearly independent set. Hence the knowledge of $\alpha(A)$ is sufficient to find A.

Also solved by Josef Andersson, Vaxholm, Sweden; J. L. Brown, Jr., Pennsylvania State University; L. Carlitz, Duke University; Daniel I. A. Cohen, Princeton University; Jack C. Gbad, University of San Francisco; B. A. Hausmann, West Baden College, Indiana; Harry W. Hickey, Alexandria, Virginia; Gilbert Labelle, Université de Montréal, Canada; Earl H. McKinney, Ball State Teachers College; P. R. Nolan, Department of Education, Dublin, Ireland; Michael J. Pascual, Watervliet Arsenal, New York; David L. Silverman, Beverly Hills, California; and the proposer.

Pick-a-Point

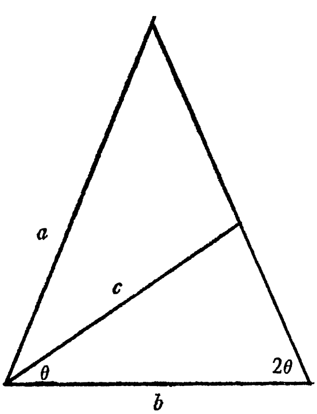
519. [May 1963] *Proposed by Michael J. Pascual, Watervliet Arsenal, New York.*

A point is chosen at random on a line segment of length l. What is the probability that an isosceles triangle can be formed by using either segment as the base and the remaining segment as the bisector of the base angle?

Solution by Ben K. Gold, Los Angeles City College.

In the figure, $0<\theta<\pi/4$ and the ratio R is

$$R=b/c=\frac{\sin(\pi-3\theta)}{\sin 2\theta}=\frac{\sin 3\theta}{2\sin\theta\cos\theta}=\frac{3\sin\theta-4\sin^3\theta}{2\sin\theta\cos\theta}=\frac{3-4\sin^2\theta}{2\cos\theta}.$$



If $\theta=0$, $R=3/2$, while if $\theta=\pi/4$, $R=1/\sqrt{2}$; thus $1/\sqrt{2}<R<3/2$.

Now let the base be of length 1 and let x be the point of division. For a triangle to be possible under the given conditions, either $1/\sqrt{2}<x/(1-x)<3/2$, or $1/\sqrt{2}<(1-x)/x<3/2$. That is, either $\sqrt{2}-1<x<3/5$, or $2/5<x<2-\sqrt{2}$. Combining these inequalities leads to $2/5<x<3/5$. Thus the probability is $1/5$.

Also solved by Josef Andersson, Vaxholm, Sweden, and the proposer. Three incorrect solutions were received.

Puzzled Prodigy

520. [May 1963] *Proposed by L. S. Shively, Ball State Teachers College.*

A prodigy gave 1.94608974115866 as the value of one of the roots, to fifteen digits, of the equation

$$x^9 - 9x^7 + 27x^5 - 30x^3 + 9x + 1 = 0.$$

A certain thirteen of the fifteen digits are correct and the other two are incorrect. Which ones are incorrect, and what is the value of this root, correct to fifteen digits?

Solution by Dermott A. Breault, Sylvania Applied Research Laboratory, Waltham, Massachusetts.

1. The given polynomial may be rewritten in the form:

$$F(s) = s^3 - 3s + 1, \tag{1}$$

with

$$s = x^3 - 3x. \tag{2}$$

2. Using Cardan's formula, the roots of (1) are found to be 2 cos 40°, 2 cos 160°, and 2 cos 280°.

3. The given root, x, is a solution of (2), with s replaced by one of these roots of (1). Since $x(x^2-3)$ is positive near $x=1.95$, we reject the second of those three roots, since it is negative.

4. Choose $s=2\cos 40°$ in (2), and we must now solve:

$$x^3 - 3x - 2\cos 40° = 0. \tag{3}$$

5. The substitution $x=2\cos\phi$ leads to:

$$4\cos^3\phi - 3\cos\phi - \cos 40° = 0, \tag{4}$$

so

$$\cos 3\phi = \cos 40°, \tag{5}$$

and

$$\phi = 13\ 1/3°, \qquad x = 2\cos(13\ 1/3°). \tag{6}$$

6. x was found using the NBS Table of Sines and Cosines to 15D at Hundredths of a Degree (AMS 5). The recommended Everett formula was used to

keep all 15 places, to get:

$$.5x = .973044870579823$$

so

$$x = 1.94608974115965,$$

and the 13th and 15th digits of the proposed root should be changed from 8 and 6, to 9 and 5 respectively.

Also solved by the proposer. One partial solution was received.

A Perfect Square Sum

521. [May 1963] *Proposed by Leo Moser, University of Alberta.*

Prove that

$$\sum_{i=1}^{2n-1} 2^{i-1}\binom{4n-2}{2i}$$

is a perfect square for $n=1, 2, 3, \cdots$.

Solution by Henry W. Gould, West Virginia University.

To prove that

$$\sum_{k=1}^{2n-1}\binom{4n-2}{2k}2^{k-1} \tag{1}$$

is a perfect square for $n=1, 2, 3, \cdots$, we may proceed as follows.

From the binomial expansion of $(1+x)^m$ we have, by separating the even and odd index terms, the general closed formula

$$\sum_{k=0}^{[m/2]}\binom{m}{2k}x^k = \frac{(1+\sqrt{x})^m+(1-\sqrt{x})^m}{2}. \tag{2}$$

Setting $m=4n-2$ and $x=2$, we find after a little simplification

$$\sum_{k=1}^{2n-1}\binom{4n-2}{2k}2^{k-1} = \frac{(1+\sqrt{2})^{4n-2}+(1-\sqrt{2})^{4n-2}-2}{4}. \tag{3}$$

Since in fact

$$\left\{(1+\sqrt{2})^{2n-1}+(1-\sqrt{2})^{2n-1}\right\}^2$$
$$= (1+\sqrt{2})^{4n-2}+(1-\sqrt{2})^{4n-2}-2;$$

the desired conclusion follows at once. The expression (2) defines a kind of generalized Fibonacci or Lucas number, we might observe.

Also solved by Josef Andersson, Vaxholm, Sweden; W. J. Blundon, Memorial University of Newfoundland; L. Carlitz, Duke University; Jack C. Gbad, University of San Francisco; Harry M. Gehman, SUNY at Buffalo, New York; Eldon Hansen, Lockheed Missiles and Space Co., Palo Alto, California; John M. Howell, Los Angeles City College; Gilbert Labelle, Université de Montréal, Canada; Francis D. Parker, University of Alaska; and Boris Pavković, Zagreb, Yugoslavia.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q327. Determine all the triangles such that

$$a^2 + b^2 - 2ab\lambda \cos C = b^2 + c^2 - 2bc\lambda \cos A$$
$$= c^2 + a^2 - 2ca\lambda \cos B$$

[*Submitted by Murray S. Klamkin.*]

Q328. Determine

$$\int_0^\infty \frac{1 - e^{-t}}{t^m}\, dt \quad \text{where} \quad (2 > m > 1).$$

[*Submitted by Murray S. Klamkin.*]

Q329. Prove that for n a positive integer $n^4+2n^3+2n^2+2n+1$ is never a perfect square. [*Submitted by Leo Moser.*]

Q330. Factor the polynomial

$$P(x) = 1 - \binom{x}{1} + \binom{x}{2} - \cdots (-1)^n \binom{x}{n}, \qquad n = 1, 2, 3 \cdots.$$

[*Submitted by Josef Andersson.*]

(Answers on page 53)

FALSIES

A falsie is a problem for which a correct solution is obtained by illegal operations, or an incorrect result is secured by apparently legal processes. For each of the following falsies, can you offer an explanation? Send us your favorite falsies for publication.

F20. To simplify $\sqrt{3\frac{3}{8}}$, the whole number is taken from under the radical, so $\sqrt{3\frac{3}{8}} = 3\sqrt{\frac{3}{8}}$. [*Submitted by C. W. Trigg.*]

F21. Integrating by parts

$$\int \cot x dx = \int \csc x d(\sin x)$$

$$= \csc x \sin x - \int \sin x d(\csc x)$$
$$= 1 + \int \cot x dx.$$

Hence $0 = 1$. [*Submitted by C. F. Pinzka.*]

(Explanations on page 53)

E20. If

$$\sqrt{\left\{a+\frac{b}{c}\right\}}=a\sqrt{\frac{b}{c}},$$

then $ac+b=a^2b$ or $ac=b(a^2-1)$. Now a is prime to a^2-1, so divides b and a^2-1 divides c. This problem is a special case in which $a=3$, $b=3$, $c=8=a^2-1$. Other mixed numbers in this same curious category, $a+a/(a^2-1)$, are $2\frac{2}{3}$, $4\frac{4}{15}$, $5\frac{5}{24}$, $6\frac{6}{35}$, etc.

E21. One explanation is that when an indefinite integral occurs, the equality is to be regarded as an equivalence relation such that two functions are equivalent when they differ by a constant.

(Falsies on page 62)

Answers

A327. $a^2+b^2-2ab\lambda \cos C=(1-\lambda)a^2+(1-\lambda)b^2+\lambda c^2$. Consequently, $a=b=c$ unless $\lambda=1/2$ for which case the equations are identically satisfied.

A328. Let

$$\phi(a)=\int_0^\infty \frac{1-e^{-at}}{t^m}\,dt$$

then

$$\phi'(a)=\int_0^\infty \frac{e^{-at}}{t^{m-1}}\,dt=\frac{1}{a^{2-m}}\,\Gamma(2-m).$$

Hence

$$\phi(1)=\frac{1}{m-1}\,\Gamma(2-m)=-\,\Gamma(1-m).$$

This procedure can be extended to the integrals of the form

$$\int_0^\infty\left(1-t+\frac{t^2}{2!}-\cdots-e^{-t}\right)\frac{dt}{t^r}\,.$$

A329. The given number lies between the squares of

$$n^2+n \quad\text{and}\quad n^2+n+1.$$

A330. We have

$$\begin{aligned}0=(1-1)^r&=\sum_{s=0}^{r}(-1)^s\binom{r}{s}\\&=\sum_{s=0}^{n}(-1)^s\binom{r}{s}, \qquad r=1,2,\cdots,n.\end{aligned}$$

Therefore $P(x)=(1-x)(2-x)\cdots(n-x)/n!$

(Quickies on page 62)

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